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ALGEBRAS GROUPS AND GEOMETRIES

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A TRIALITY GROUP OF NONASSOCIATIVE ALGEBRAS WITH INVOLUTION

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Abstract

In this paper, we define the concept of a triality group and consider the group associated with algebras. In particular, we study a characterization of this group for symmetric composition algebras and other nonassociative algebras.

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Introduction

The main purpose of this study is to propose a generalization of the automorphism group associated with nonassociative algebras with involution. Moreover, we will provide a new idea or a frontier concept of a triality relation for the group called a global triality relation.

First, let A be a nonassociative algebra over a field F. If a triple $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in (Epi(A))^3$, where $Epi(A)$ is the set of onto endomorphisms of A, satisfies $\sigma_j(xy) = (\sigma_{j+1}x)(\sigma_{j+2}y)$ and $\sigma_j(x+y) = \sigma_j x + \sigma_j y$ for $j = 1, 2, 3$ with $\sigma_{j\pm 3} = \sigma_j$ and for any $x, y \in A$, then the set of all such triples forms a group called a triality group of A under componentwise multiplication and the triality group is denoted by $Trig(A)$.

In particular, we consider an explicit construction of the triality group of symmetric composition algebras A and try to find subgroups and invariant subgroups of $Trig(A)$ (see Theorem 2.5). By taking a binary product of left and right operators of A, we have found a formula for invariant subgroups of automorphism groups of symmetric composition algebras (see Theorem 4.1 and Corollary 4.2).

Summarizing this article, in the following sections, we will discuss

1. triality groups (definitions and preliminaries),
2. symmetric composition algebras,
3. local triality relations,
4. automorphisms of symmetric composition algebras,
5. examples of triality groups for some nonassociative algebras with involution.

Some simple examples are also given in the appendix.

This article may be viewed as a continuation of our previous paper ([K-O.15]). However the content is independent of previous work, that is, we have made this paper as self-contained as possible. If further background to the subjects adreseed in this article is needed, for algebras, please refer to ([A.78],[A-F.93],[E.97],[E.00],[J.58],[K-M-P-T.98],[K-O.14],[O.95],[O.05],[O-O.81],[S.66]) and furthermore, for ternary algebras, please refer to ([K-M-O.10],[K-O.00],[K-O.14],[K-O.16]).

Finally, we would like to note that this work was written at the end of the life of one of the authors (S. Okubo, 1930-2015).

1 Triality Groups (Definitions and Preliminaries)

Let A be an algebra over a field F of characteristic not 2 with a bilinear product denoted by juxtaposition xy for $x, y \in A$. Suppose that a triple $g = (g_1, g_2, g_3) \in (EpiA)^3$, where $EpiA$ denotes the set of epimorphisms of A, satisfies the global triality relation

$$g_j(xy) = (g_{j+1}x)(g_{j+2}y) \tag{1.1}$$

for any $x, y \in A$ and for any $j = 1, 2, 3$. Here the index j is defined by modulo 3, i.e.,

$$g_{j\pm 3} = g_j. \tag{1.2}$$

For a second triple $g' = (g_1', g_2', g_3') \in (EpiA)^3$ satisfying the same triality relation, we introduce their componentwise product as

$$gg' = (g_1 g_1', g_2 g_2', g_3 g_3'). \tag{1.3}$$

Hence, the set consisting of all such triples forms a group, then we call a "*triality group*" of A, and we write

$$Trig(A) = \{g = (g_1, g_2, g_3) \in (EpiA)^3 |$$

$$g_j(xy) = (g_{j+1}x)(g_{j+2}y), \forall x, y \in A, \ \forall\, j = 1, 2, 3\}. \tag{1.4}$$

Here we emphasize that this group is clearly a generalization of the automorphism group defined by

$$Auto(A) = \{g \in EpiA | g(xy) = (gx)(gy), \forall x, y \in A\}. \tag{1.5}$$

We then note that $Trig(A)$ is invariant under actions of the alternating group A_4 (or equivalently the tetrahedral group) as follows. First, let $\phi \in End(\mathrm{Trig}(A))$ be given by

$$\phi : g_1 \to g_2 \to g_3 \to g_1, \tag{1.6}$$

which satisfies $\phi^3 = id$ and leaves Eq.(1.1) invariant. Thus, $Trig(A)$ is invariant under actions of the cyclic group Z_3 generated by ϕ. We next introduce $\tau_\mu \in End(Trig(A))$ for $\mu = 1, 2, 3$ by

$$\tau_1 : g_1 \to g_1, \ g_2 \to -g_2, \ g_3 \to -g_3$$

$$\tau_2 : g_1 \to -g_1,\ g_2 \to g_2,\ g_3 \to -g_3 \tag{1.7}$$
$$\tau_3 : g_1 \to -g_1,\ g_2 \to -g_2,\ g_3 \to g_3,$$

which leave Eq.(1.1) invariant again. Moreover, they satisfy

$$\tau_\mu \tau_\nu = \tau_\nu \tau_\mu,\ \tau_\mu^2 = id,\ \tau_1\tau_2\tau_3 = id \tag{1.8}$$

for $\mu, \nu = 1, 2, 3$; thus, the group generated by $< id, \tau_1, \tau_2, \tau_3 >$ is isomorphic to the Klein 4-group K_4.

Further, we note that

$$\phi \tau_\mu \phi^{-1} = \tau_{\mu+1} \text{ (with } \tau_4 = \tau_1). \tag{1.9}$$

Since Z_3 and K_4 generate the alternating group A_4, this shows that A_4 is an invariant group of $Trig(A)$, for example, $(\tau_1 g_1, \tau_1 g_2, \tau_1 g_3) = (g_1, -g_2, -g_3) \in Trig(A)$.

If A is involutive with the involution map $x \to \bar{x}$ satisfying

$$\bar{\bar{x}} = x,\ \overline{xy} = \bar{y}\bar{x}, \tag{1.10}$$

we define $\bar{Q} \in End(A)$ for any $Q \in End(A)$ by

$$\overline{Qx} = \bar{Q}\bar{x}. \tag{1.11}$$

then, taking the involution of Eq.(1.10) and letting $x \leftrightarrow \bar{y}$, we find

$$\bar{g}_j(xy) = (\bar{g}_{j+2}x)(\bar{g}_{j+1}y), \tag{1.12}$$

implying that $\theta \in End(Trig(A))$ defined by

$$\theta : g_1 \to \bar{g}_2,\ g_2 \to \bar{g}_1,\ g_3 \to \bar{g}_3 \tag{1.13}$$

also yields an invariant operation of $Trig(A)$. Moreover, we obtain

$$\phi\theta\phi = \theta,\ \theta^2 = id,\ \theta\tau_1\theta^{-1} = \tau_2,\ \theta\tau_2\theta^{-1} = \tau_1,\ \theta\tau_3\theta^{-1} = \tau_3. \tag{1.14}$$

Then, A_4 together with θ give the S_4-symmetry for $Trig(A)$ with the identification of

$$\phi = (1,2,3),\ \tau_1 = (2,3)(1,4), \tau_2 = (3.1)(2.4), \tau_3 = (1,2)(3,4), \theta = (1,2) \tag{1.15}$$

in terms of the transpositions of the S_4-group. This also shows that S_4 is an invariant group of $Trig(A)$ if A is involutive.

Before going into further discussion, we note that $Trig(A)$ for any A always contains a Klein 4-group K_4. In fact, let $Id \in Epi(A)$ be defined by $(Id)x = x$ for $x \in A$ and put

$$\tilde{\tau}_0 = (Id, Id, Id),\ \tilde{\tau}_1 = (Id. - Id, -Id),$$

$$\tilde{\tau}_2 = (-Id, Id, -Id),\ \tilde{\tau}_3 = (-Id, -Id, Id). \tag{1.16}$$

Then, these four elements of $(Epi(A)^3)$ satisfy Eq.(1.8) in view of Eq.(1.3); thus, they give another Klein 4-group. Moreover, they satisfy Eq.(1.1). Similarly, we have $Trig(F) \cong K_4$ for the simplest case of $A = F$(a field).

In contrast to the global triality relation Eq.(1.1), we may also consider the local triality relation

$$t_j(xy) = (t_{j+1}x)y + x(t_{j+2}y) \tag{1.17}$$

for $t_j \in End(A)$ with $t_{j\pm 3} = t_j$. Analogously to Eq.(1.4), we introduce

$$s \circ Lrt(A) =$$

$$\{t = (t_1, t_2, t_3) \in (End\ A)^3 |$$

$$t_j(xy) = (t_{j+1}x)y + x(t_{j+2}y), \forall x, y \in A, \forall i = 1, 2, 3\}. \tag{1.18}$$

Then, the set of $s \circ Lrt(A)$ now defines a Lie algebra with a componentwise commutation relation ([K-O.15]).

Here, we use $s \circ Lrt(A)$ to denote a symmetric Lie-related triple, which has been rewritten as $stri(A)$ in [O.05].

To see this symmetric Lie-related triple, if we set

$$t_j' = \sum_{k=1}^{3} \alpha_{j-k} t_k,\ (j = 1, 2, 3) \tag{1.19}$$

for any $\alpha_j \in F$ satisfying $\alpha_{j\pm 3} = \alpha_j$, then it is easy to verify that we also have

$$t' = (t_1', t_2', t_3') \in s \circ Lrt(A). \tag{1.20}$$

If the exponential map $t_j \to \xi_j$ given by

$$\xi_j = \exp(t_j) = \sum_{n=0}^{\infty} \frac{1}{n!}(t_j)^n \tag{1.21}$$

is well defined, then we have shown in [K-O.15] the validity of

$$\xi_j(xy) = (\xi_{j+1}x)(\xi_{j+2}y) \tag{1.22}$$

provided that $t = (t_1, t_2, t_3) \in s \circ Lrt(A)$ and vice versa.

Note that the existence of the exponential map requires the underlying field F to be at least of zero characteristic.

We first introduce multiplication operators $L(x), R(x) \in End(A)$ as

$$L(x)y = xy, \ R(x)y = yx \tag{1.23}$$

as usual. Then, Eq.(1.17) yields

$$t_j L(x) = L(x)t_{j+2} + L(t_{j+1}x) \tag{1.24a}$$

$$t_j R(y) = R(y)t_{j+1} + R(t_{j+2}y), \tag{1.24b}$$

while Eq.(1.1) gives

$$g_j L(x) = L(g_{j+1}x)g_{j+2} \tag{1.25a}$$

$$g_j R(y) = R(g_{j+2}y)g_{j+1} \tag{1.25b}$$

for any $g = (g_1, g_2, g_3) \in Trig(A)$.

From Eqs.(1.25), we find

$$g_j L(x)R(y)g_j^{-1} = L(g_{j+1}x)R(g_{j+1}y) \tag{1.26a}$$

$$g_j R(y)L(x)g_j^{-1} = R(g_{j+2}y)L(g_{j+2}x). \tag{1.26b}$$

We next introduce the notion of a regular triality algebra.

Def.1.1

Let $d_j(x, y) \in End(A)$ for $x, y \in A$ and for $j = 1, 2, 3$ satisfy
(i)

$$d_1(x, y) = R(y)L(x) - R(x)L(y) \tag{1.27a}$$

$$d_2(x,y) = L(y)R(x) - L(x)R(y); \tag{1.27b}$$

(ii) the explicit form for $d_3(x,y)$ is unspecified except for

$$d_3(y,x) = -d_3(x,y); \tag{1.27c}$$

(iii) $(d_1(x,y), d_2(x,y), d_3(x,y)) \in s \circ Lrt(A)$, i.e., they satisfy

$$d_j(x,y)(uv) = (d_{j+1}(x,y)u)v + u(d_{j+2}(x,y)v) \tag{1.28}$$

for any $x,y,u,v \in A$ and for any $j = 1,2,3$. Here the index over j is defined modulo 3 as before.

We call the algebra A satisfying these conditions a regular triality algebra ([K-O.15]).

For the case of Lie algebra A equipped with the product $[x,y]$ as usual, if we set

$$d_1(x,y) = R(y)L(x) - R(x)L(y)$$
$$d_2(x,y) = L(y)R(x) - L(x)R(y)$$
$$d_3(x,y) = d_0(x.y) = L([x,y]),$$

where $L(x)y = [x,y]$ and $R(x)y = [y,x]$, then we obtain

$$d_j = L([x,y]) \text{ for any } j = 0,1,2.$$

Thus, this implies that d_j is an inner derivation of A (i.e., a special example of Def.1.1).

Def.1.2

Condition (B): We have $AA = A$.

Condition (C): If some $b \in A$ satisfies either $L(b) = 0$ or $R(b) = 0$, then $b = 0$.

We can now prove the following.

Proposition 1.3

Let A be a regular triality algebra satisfying either condition (B) or (C). We then obtain the following:

(i) For any $t = (t_1, t_2, t_3) \in s \circ Lrt(A)$, *we have*

$$[t_j, d_k(x,y)] = d_k(t_{j-k}x, y) + d_k(x, t_{j-k}y). \tag{1.29a}$$

In particular, choosing $t_j = d_j(u,v)$ *also yields*

$$[d_j(u,v), d_k(x,y)] = d_k(d_{j-k}(u,v)x, y) + d_k(x, d_{j-k}(u,v)y). \tag{1.29b}$$

(ii) For any $g = (g_1, g_2, g_3) \in Trig(A)$, *we have*

$$g_j d_k(x,y) g_j^{-1} = d_k(g_{j-k}x, y) + d_k(x, g_{j-k}y) \tag{1.30}$$

for any $j, k = 1, 2, 3$ *and for any* $u, v, x, y \in A$.

Proof

Since Eqs.(1.29) have already been proved in [K-O.15], we will only give a proof of Eq.(1.30). In view of Eqs.(1.26) and (1.27), we see that Eq.(1.30) holds for any $j = 1, 2, 3$ and for $k = 1, 2$. Therefore, it suffices to prove the case of $k = 3$. To this end, we set

$$D_{j,k} :\equiv g_j d_k(x,y) g_j^{-1} - d_k(g_{j-k}x, y) - d_k(x, g_{j-k}y) \tag{1.31}$$

for a fixed $x, y \in A$. Then, as we have noted, we have $D_{j,1} = D_{j,2} = 0$ identically. Moreover, we will show that it also satisfies

$$D_{j,k}(uv) = (D_{j+1,k+1}u)v + u(D_{j+2,k+2}v). \tag{1.32}$$

We first calculate

$$g_j d_k(x,y) g_j^{-1}(uv) = g_j d_k(x,y)\{(g_{j+1}^{-1}u)(g_{j+2}^{-1}v)\} =$$

$$g_j(\{d_{k+1}(x,y)(g_{j+1}^{-1}u)\}(g_{j+2}^{-1}v)) + g_j((g_{j+1}^{-1}u)(d_{k+2}(x,y)g_{j+2}^{-1}v))$$

$$= g_{j+1}(d_{k+1}(x,y)(g_{j+1}^{-1}u))g_{j+2}(g_{j+2}^{-1}v)) + (g_{j+1}g_{j+1}^{-1}u)(g_{j+2}d_{k+2}(x,y)g_{j+2}^{-1}v)$$

$$= g_{j+1}(d_{k+1}(x,y)g_{j+1}^{-1}u)v + u(g_{j+2}d_{k+2}(x,y)g_{j+2}^{-1}v).$$

Similarly, we find

$$\{d_k(g_{j-k}x, y) + d_k(x, g_{j-k}y)\}(uv) =$$

$$\{d_{k+1}(g_{j-k}x,y)u\}v + u\{d_{k+2}(d_{j-k}x,y)v\}$$
$$+\{d_{k+1}(x,g_{j-k}u\}v + u\{d_{k+2}(x,g_{j-k}y)v\},$$

and this proves the validity of Eq.(1.32) since

$$g_j d_k(x,y)g_j^{-1}(uv) - d_k(x,g_{j-k}y)(uv) - d_k(g_{j-k}x,y)(uv) =$$
$$(D_{j+1,k+1}u)v + u(D_{j+2.j*2}v).$$

If we choose $k = 3$ in Eq.(1.32), we obtain

$$D_{j,3}(uv) = (D_{j+1,1}u)v + u(D_{j+2,2}v) = 0,$$

which gives $D_{j,3} = 0$, provided that condition (B) holds.
On the other hand, the choice of $k = 1$ or $k = 2$ in Eq.(1.32) yields

$$0 = u(D_{j+2,3}v) = (D_{j+1,3}u)v$$

for any $j = 1, 2, 3$ and for any $u, v \in A$. Therefore, it also gives $D_{j,3} = 0$ under condition (C). This completes the proof.□

We are now in a position to construct an invariant subgroup of $Trig(A)$ as follows:

Let A be a regular triality algebra satisfying either condition (B) or (C), and set

$$L_0 = span < d_j(x,y), \forall x, y \in A, \forall j = 1,2,3 > . \tag{1.33}$$

Then L_0 is a Lie algebra by Eq.(1.29b). Moreover, it is an ideal of the larger Lie algebra $s \circ Lrt(A)$ by Eq.(1.29a). For any basis $e_1, e_2 \cdots, e_N$ of A with $N = Dim\ A$, and for any $\alpha_{j,\mu,\nu} \in F(j = 1,2,3,\ \mu,\ \nu = 1,2,\cdots,N)$, we set

$$D_j = \sum_{k=1}^{3}\sum_{\mu,\nu=1}^{N} \alpha_{j-k,\mu,\nu} d_k(e_\mu, e_\nu) \in L_0 \tag{1.34}$$

for $j = 1,2,3$. Then, $D = (D_1, D_2, D_3)$ is a member of $s \circ Lrt(A)$ by Eq.(1.29). Therefore, its exponential map

$$\xi_j = exp\ D_j\ (j = 1,2,3) \tag{1.35}$$

satisfies $\xi_j(xy) = (\xi_{j+1}x)(\xi_{j+2}y)$, i.e., $(\xi_1, \xi_2, \xi_3) \in Trig(A)$, provided that the exponential map is well-defined. Moreover, for any $g = (g_1, g_2, g_3) \in Trig(A)$, we calculate

$$g_j D_k g_j^{-1} = \sum_{l=1}^{3} \sum_{\mu,\nu=1}^{N} \alpha_{k=l,\mu,\nu} \{ d_l(g_{j-l}e_\mu, e_\nu) + d_j(e_\mu, g_{j-l}e_\nu) \} \in L_0 \quad (1.36)$$

by Eq.(1.30), implying that

$$g_j \xi_k g_j^{-1} = g_j(exp\ D_k) g_j^{-1} = exp(g_j D_k g_j^{-1}) \in exp\ L_0. \quad (1.37)$$

Therefore, the group G_0 generated by $< \xi_1, \xi_2, \xi_3 >$ is an invariant subgroup of $Trig(A)$.

Remark 1.4

If a regular triality algebra satisfies Eqs.(1.27) as well as

$$d_3(x, y)z + d_3(y, z)x + d_3(z, x)y = 0, \quad (1.38)$$

then A is known as a pre-normal triality algebra ([K-O.15]). Moreover, if we have

$$Q(x, y, z) = d_1(z, xy) + d_2(y, zx) + d_3(x, yz) = 0 \quad (1.39)$$

in addition, A is called a normal triality algebra in [K-O.15].

Further, suppose that A is involutive with the involution map $x \to \bar{x}$, as in Eq.(1.10), and we introduce the second bilinear product in the same vector space of A as

$$x * y = \overline{xy} = \bar{y}\bar{x}. \quad (1.40)$$

Then the resulting algebra A^* is also involutive; this algebra A^* is called a conjugation algebra of A. Furthermore, we note that a structurable algebra, introduced by Allison ([A.78],[A-F.93]), is a class of the conjugation algebra A^* of the normal triality algebra A, and this nonassociative algebra A^* contains associative, commutative Jordan, and alternative algebras ([K-O.15]).

For a Lie algebra $(A,\ [\ ,\ ])$, setting $\bar{x} = -x$, from $x * y = \overline{[x,y]}$, we have

$$\overline{[x,y]} = -[x,y] = [y,x] = [\bar{y},\bar{x}],$$
$$\overline{x*y} = [x,y] = \overline{[\bar{y},\bar{x}]} = \bar{y} * \bar{x}.$$

This implies that A and A^* are involutive with respect to $\bar{x} = -x$.

If A^* is unital (i.e., there is an element $e \in A^*$ such that $e*x = x*e = x$ for any $x \in A^*$) and A is a normal triality algebra, then this algebra A^* is structurable and both conditions (B) and (C) are automatically satisfied (see [K-O.14]).

Finally, Eq.(1.1) is rewritten as the modified global triality relation

$$\bar{g}_j(x * y) = (g_{j+1}x) * (g_{j+2}y). \tag{1.41}$$

Reviewing these results, we can construct an invariant subgroup of $Trig(A)$ from any regular triality algebra A satisfying condition (B) or (C), in particular, from any structurable algebra A^*. However, for some cases, we can more directly construct an invariant subgroup of $Trig(A)$, as for the case of A being a symmetric composition algebra. The case will be explored in the next section, while we will study its relation to the corresponding $s \circ Lrt(A)$ in section 3. If we further restrict ourselves to the automorphism group, then we can also construct an explicit automorphism of the Cayley algebra, which we will study in section 4. Finally, other cases of $Trig(A)$ for several algebras A with involution will be given in section 5.

2 Symmetric Composition Algebras

This section is devoted to constructions of $Trig(A)$ for symmetric composition algebras. Let A be an algebra over a field F of characteristic not 2 with a symmetric bilinear nondegenerate form $< \cdot|\cdot >$. Suppose that it satisfies

$$(xy)x = x(yx) =< x|x > y \tag{2.1}$$

for any $x, y \in A$. Then A is called a symmetric composition algebra since it also satisfies the composition law

$$< xy|xy >=< x|x >< y|y > \text{ and } < xy|z >=< x|yz > . \tag{2.2}$$

Conversely, the validity of Eq.(2.2) implies that of Eq.(2.1) ([O.95]).

Linearizing Eq.(2.1) gives

$$(xy)z + (zy)x = x(yz) + z(yx) = 2 < x|z > y \tag{2.3}$$

for any $x, y, z \in A$. Moreover, replacing z by yz in the first relation of Eq.(2.3) and noting Eq.(2.2) yields

$$(xy)(yz) = 2 < x|yz > y - < y|y > zx. \tag{2.4}$$

Also, for any $g = (g_1, g_2, g_3) \in Trig(A)$, we have

$$< g_j x|g_j y > = < x|y > \ (j = 1, 2, 3) \tag{2.5}$$

for the following reason. Applying g_j to both sides of Eq.(2.1) gives

$$< x|x > g_j y = g_j\{x(yx)\} = (g_{j+1}x)(g_{j+2}(yx)) =$$

$$(g_{j+1}x)\{(g_j y)(g_{j+1}x)\} = < g_{j+1}x|g_{j+1}x > g_j y,$$

implying that we have $< x|x > = < g_{j+1}x|g_{j+1}x >$. Linearizing the relation and letting $j \to j - 1$, we obtain Eq.(2.5). Similarly, any $t_j \in s \circ Lrt(A)$ satisfies

$$< t_j x|y > = - < x|t_j y > \ (j = 1, 2, 3) \tag{2.6}$$

from the relations $t_j(x(yx)) = < x|x > t_j(y)$ and $< x|t_j(x) > = 0$.

Any symmetric composition algebra over the field F of characteristic not 2 is known (see [O-O.81],[E.97],[K-M-P-T.98], or Theorem 2 of section 4.3 in ([O.95]) to be either a para-Hurwitz algebra or an 8-dimensional pseudo-octonion algebra, where the para-Hurwitz algebra is the conjugate algebra of the Hurwitz algebra with the para-unit e satisfying $ex = xe = \bar{x} = 2 < e|x > e - x$. That is, we emphasize that quaternion and Cayley algebras with the product $x * y$ have the structure of a symmetric composition algebra with respect to the new product $xy = \overline{x * y}$, where $\bar{x}$ is the standard conjugation of x. Moreover, the symmetric composition algebra is a normal triality algebra ([O.05]). Further, conditions (B) and (C) of section 1 are automatically satisfied by this

algebra. Since $< o|o >$ is assumed to be nondegenerate, there exists $x \in A$ satisfying $< x|x > \neq 0$. Now Eq.(2.1) implies $< x|x > y \in AA$ for any element $y \in A$; thus, we have $AA = A$. If $b \in A$ satisfies $bA = 0$, then Eq.(2.1) also leads to $< x|x > b = 0$, and here $b = 0$. Therefore, both conditions (B) and (C) are automatically satisfied. In particular, any symmetric composition algebra is a regular triality algebra satisfying both conditions (B) and (C) (see Remark 1.4). Thus, we can construct an invariant subgroup of $Trig(A)$, as has been demonstrated in the previous section. However, for the special cases of $Dim\ A = 1$ and $Dim\ A = 2$, we can construct the entire $Trig(A)$ as follows. For the base field F, which is given in Examples 2.1 and 2.2, to form an orthonormal basis e and f satisfying $< e|e >= 1$ and $< f|f >= 1$, we assume that if any $\alpha^2 \in F$ then $\alpha \in F$.

Example 2.1. Case of $Dim\ A = 1$

We write $A = Fe$, where $e \in A$ satisfies $ee = e$ with $< e|e >= 1$. Then for any $g = (g_1, g_2, g_3) \in \mathrm{Trig}(A)$, we can write

$$g_j(e) = \alpha_j e \ (j = 1, 2, 3)$$

for some $\alpha_j = \alpha_{j+1}\alpha_{j+2}$, while Eq.(2.5) yields $(\alpha_j)^2 = 1$. Therefore, this gives

$$\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = 1 \text{ and } \alpha_1\alpha_2\alpha_3 = 1,$$

implying that $\mathrm{Trig}(Fe)$ is isomorphic to the Klein 4-group K_4. Note that we have $\mathrm{Auto}(Fe) =< Id >$ being trivial.

Example 2.2. Case of $Dim\ A = 2$

In this case, $A = \mathrm{span} < e, f >$, where $e, f \in A$ satisfy

$$ee = e,\ ff = -e,\ ef = fe = -f \tag{2.7a}$$

$$< e|e >=< f|f >= 1,\ < e|f >=< f|e >= 0. \tag{2.7b}$$

Note that A is Abelian, and any $x, y \in A$ satisfy the quadratic relation

$$xy = yx = - < e|x > y - < e|y > x + (4 < e|x >< e|y > - < x|y >)e. \tag{2.8}$$

Note also that e is the para-unit of A, i.e.,

$$ex = xe = \bar{x} = 2 < e|x > e - x. \tag{2.9}$$

Let us write

$$p_j = g_j(e), q_j = g_j(f), \ (j = 1, 2, 3). \tag{2.10}$$

Then, the global triality relation Eq.(1.1) is equivalent to the validity of

$$p_{j+1}p_{j+2} = p_j, \ p_{j+1}q_{j+2} = q_{j+1}p_{j+2} = -q_j, \ q_{j+1}q_{j+2} = -p_j \tag{2.11}$$

for $j = 1, 2, 3$, where the indices over j are defined modulo 3, i.e., $p_{j\pm3} = p_j, q_{j\pm3} = q_j$. Also, Eqs.(2.5) and (2.7*b*) give

$$< p_j|p_j >=< q_j|q_j >= 1, \ < p_j|q_j >= 0 \ (j = 1, 2, 3). \tag{2.12}$$

The general solution of Eqs.(2.11) and (2.12) is given by the following proposition.

Proposition 2.3

Let $q_j \in A$ for $j = 1, 2, 3$ be any three elements of A satisfying

$$< q_1|q_1 >=< q_2|q_2 >=< q_3|q_3 >= 1, \text{ and } < q_3|q_1q_2 >= 0. \tag{2.13}$$

If we introduce p_j as

$$p_j = -q_{j+1}q_{j+2}, \tag{2.14}$$

then this p_j satisfies Eqs.(2.11) and (2.12). Conversely, if p_j and q_j satisfy Eqs.(2.11) and (2.12), then Eq.(2.13) holds.

Proof

We first note that $< q_3|q_1q_2 >= 0$ also implies the validity of

$$< q_j|q_{j+1}q_{j+2} >= 0 \tag{2.15}$$

for all $j = 1, 2, 3$ because of the associative law $< xy|z >=< x|yz >$.

Suppose now that Eqs.(2.13) and (2.14) hold. We then calculate

$$p_{j+1}q_{j+2} = -(q_{j+2}q_j)q_{j+2} = - < q_{j+2}|q_{j+2} > q_j = -q_j$$

$$q_{j+1}p_{j+2} = -q_{j+1}(q_jq_{j+1}) = - < q_{j+1}|q_{j+1} > q_j = -q_j.$$

Finally, we compute

$$p_{j+1}p_{j+2} = (q_{j+2}q_j)(q_jq_{j+1}) = 2 < q_{j+2}|q_jq_{j+1} > q_j - < q_j|q_j > q_{j+1}q_{j+2}$$

by Eq.(2.4) and we find that

$$p_{j+1}p_{j+2} = -q_{j+1}q_{j+2} = p_j$$

because of Eq.(2.14), proving Eq.(2.11). Moreover,

$$< p_j|p_j > = < q_{j+1}q_{j+2}|q_{j+1}q_{j+2} > = < q_{j+1}|q_{j+1} >< q_{j+2}|q_{j+2} > = 1$$

and

$$< q_j|p_j > = - < q_j|q_{j+1}q_{j+2} > = 0,$$

which establish Eq.(2.12). Conversely, if $< p_j|q_j >= 0$ with $p_j = -q_{j+1}q_{j+2}$, then they lead to $< q_j|q_{j+1}q_{j+2} >= 0$ or $< q_3|q_1q_2 >= 0$.□

Remark 2.4(a)

If we write

$$q_j = \mu_j e + \nu_j f \ (j = 1, 2, 3) \tag{2.16a}$$

for $\mu_j, \nu_j \in F$ $(j = 1, 2, 3)$, then Eq.(2.13) gives

$$\mu_j^2 + \nu_j^2 = 1, \ \mu_1\mu_2\mu_3 = \mu_1\nu_2\nu_3 + \mu_2\nu_3\nu_1 + \mu_3\nu_1\nu_2, \tag{2.16b}$$

while Eq.(2.14) for

$$p_j = \alpha_j e + \beta_j f \tag{2.17}$$

leads to

$$-\alpha_j = \mu_{j+1}\mu_{j+2} - \nu_{j+1}\nu_{j+2}, \tag{2.18a}$$

$$-\beta_j = -(\mu_{j+1}\nu_{j+2} + \nu_{j+1}\mu_{j+2}). \tag{2.18b}$$

In particular, Eq.(2.16) allows two independent variables in whose terms other μ_j and ν_j are determined. Thus, if the field F is either a real or complex field, then the resulting group manifold $Trig(A)$ will become a 2-dimensional real or complex manifold. However, when F is a finite field, then $Trig(A)$ will become a finite group. Also, it may be of interest to see that $\alpha_j =< e|p_j >$ and $\beta_j =< f|p_j >$ satisfy the polynomial identities

$$2\alpha_1\alpha_2\alpha_3 - (\alpha_1^2 + \alpha_2^2 + \alpha_3^2) + 1 = 0, \tag{2.19a}$$

$$(\beta_1^2 + \beta_1^2 + \beta_3^2)^2 + 4(\beta_1\beta_2\beta_3)^2 = 4\{\beta_1^2\beta_1^2 + \beta_2^2\beta_3^2 + \beta_3^2\beta_1^2\}. \tag{2.19b}$$

However, since the relevance of these identities is not clear, we will not go into detail.

Next for $s \circ Lrt(A)$, any $t = (t_1, t_2, t_3) \in s \circ Lrt(A)$ is given similarly by

$$t_j e = \lambda_j f,\ t_j f = -\lambda_j e\ (j = 1, 2, 3) \tag{2.20a}$$

for some $\lambda_j \in F$ satisfying

$$\lambda_1 + \lambda_2 + \lambda_3 = 0. \tag{2.20b}$$

Remark 2.4(b) $Aut(A)$ with $Dim\ A = 2$.

In contrast to $Trig(A)$, $Aut(A)$ for the case of $Dim\ A = 2$ turns out to always be a finite group of either Z_2 if $\sqrt{3} \notin F$ or S_3 if $\sqrt{3} \in F$. Note that $Aut(A)$ is a special case of $Trig(A)$ for $g_1 = g_2 = g_3 (= \sigma)$ with

$$\sigma(xy) = (\sigma x)(\sigma y). \tag{2.21a}$$

Setting $p_1 = p_2 = p_3 = p$ and $q_1 = q_2 = q_3 = q$, we have

$$\sigma(e) = p,\ \sigma(f) = q \tag{2.21b}$$

and Eq.(2.13) becomes

$$< q|q >= 1,\ < q|q^2 >= 0, \tag{2.22}$$

while Eq.(2.14) determines p as

$$p = -qq. \tag{2.23}$$

Setting $x = y = q$ in Eq.(2.8), we find

$$q^2 = -2 < e|q > q + (4 < e|q >< e|q > - < q|q >)e, \tag{2.24}$$

meaning that Eq.(2.22) gives

$$0 =< q|q^2 >=$$

$$-2 < e|q >< q|q > +(4 < e|q >< e|q > - < q|q >) < e|q >$$

$$=< e|q > \{4 < e|q >< e|q > -3\}, \tag{2.25}$$

whose solutions are

(i)

$$< e|q >= 0 \tag{2.26a}$$

or

(ii)

$$< e|q >= \pm\frac{\sqrt{3}}{2}, \tag{2.26b}$$

assuming $\sqrt{3} \in F$ for the second solution of Eq.(2.26b).

In terms of e and f, two cases are implied as follows.

Case 1

$\sigma : e \to e,\ f \to \epsilon f,\ (\epsilon = \pm 1),$

Case 2 (if $\sqrt{3} \in F$)

$\sigma : e \to \frac{1}{2}(-e + \sqrt{3}\epsilon_1\epsilon_2 f_1)$

$f \to \frac{1}{2}(\sqrt{3}\epsilon_1 e + \epsilon_2 f),$

where $\epsilon_1^2 = \epsilon_2^2 = 1.$

Let P and $Q \in Aut(A)$ be given by

$P : e \to e,\ f \to -f$

$Q : e \to \frac{1}{2}(-e + \sqrt{3}f), \quad f \to \frac{1}{2}(-\sqrt{3}e - f).$

Then, they satisfy

$$P^2 = Q^3 = Id,\ QPQ = P,$$

which generate the symmetric group S_3 (or equivalently the dihedral group D_3). Thus, we conclude

$$Aut(A) = \begin{cases} Z_2 =< Id, P >, \text{ if} \sqrt{3} \notin F \\ S_3 =< Id, P, Q, Q^2, PQ, PQ^2 >, \text{ if } \sqrt{3} \in F, \end{cases}$$

which are finite groups, in contrast to the case of $Trig(A)$ (at least for a real or complex field F). Correspondingly, we have $Der(A) = 0$ for the derivation assuming $3 \neq 0$.

Other cases of $Dim\ A = 4$ (para-quaternion algebra) and $Dim\ A = 8$ (either para- or pseudo-octonion algebra) are unfortunately too complicated to be discussed by a similar method. However, we can still construct some subgroups of $Trig(A)$ as follows. For any two elements $a, b \in A$ satisfying $< a|a >=< b|b >= 1$, we set $a_1 = a,\ a_2 = b$ and define $a_3 = a_1 a_2 = ab$. Then, we find

$$a_j a_{j+1} = a_{j+2},\ < a_j|a_j >= 1 \tag{2.27}$$

for any $j = 1, 2, 3$, where the indices over j are defined modulo 3, i.e.,

$$a_{j\pm 3} = a_j.$$

For example, we calculate

$$a_3 a_1 = (a_1 a_2)a_1 =< a_1|a_1 > a_2 = a_2,$$

$$< a_3|a_3 >=< a_1 a_2|a_1 a_2 >=< a_1|a_1 >< a_1|a_2 >= 1,$$

etc.

In view of this, we introduce

$$\Sigma = \{a = (a_1, a_2, a_3) \in A^3 | a_j a_{j+1} = a_{j+2},\ < a_j|a_j >= 1,\ \forall j = 1, 2, 3\}. \tag{2.28}$$

We now prove

Theorem 2.5

Let A be a symmetric composition algebra. For any $a = (a_1, a_2, a_3) \in \Sigma$, we set

$$\sigma_j(a) = R(a_{j+1})R(a_{j+2}), \tag{2.29a}$$

$$\theta_j(a) = L(a_{j+2})L(a_{j+1}) \tag{2.29b}$$

for $j = 1, 2, 3$.

Then, these relations satisfy

(i)

$$\sigma_j(a)(xy) = (\sigma_{j+1}(a)x)(\sigma_{j+2}(a)y) \tag{2.30a}$$

(ii)

$$\theta_j(a)(xy) = (\theta_{j+1}(a)x)(\theta_{j+2}(a)y) \tag{2.30b}$$

(iii)

$$\sigma_j(a)\theta_j(a) = \theta_j(a)\sigma_j(a) = Id \tag{2.30c}$$

(iv)

$$\theta_j(a)\theta_{j+1}(a)\theta_{j+2}(a) = \sigma_{j+2}(a)\sigma_{j+1}(a)\sigma_j(a) = Id \tag{2.30d}$$

(v)

$$< \sigma_j(a)x|y > = < x|\theta_j(a)y > \tag{2.30e}$$

(vi)

$$< \sigma_j(a)x|\sigma_j(a)y > = < \theta_j(a)x|\theta_j(a)y) > = < x|y > . \tag{2.30f}$$

In particular, both triples $\sigma(a) = (\sigma_1(a), \sigma_2(a), \sigma_3(a))$ *and* $\theta(a) = (\theta_1(a), \theta_2(a), \theta_3(a))$ *are elements of* $Trig(A)$.

Proof

We will first prove (iii)-(vi) as follows: Eq.(2.1) is rewritten as

$$R(x)L(x) = L(x)R(x) = < x|x > Id \tag{2.31}$$

and we calculate

$$\sigma_j(a)\theta_j(a) = R(a_{j+1})R(a_{j+2})L(a_{j+2})L(a_{j+1}) = R(a_{j+1})L(a_{j+1}) = Id$$

$$\theta_j(a)\sigma_j(a) = L(a_{j+2})L(a_{j+1})R(a_{j+1})R(a_{j+2}) = L(a_{j+2})R(a_{j+2}) = Id,$$

which prove Eq.(2.30c). We next also have

$$< \sigma_j(a)x|y > = < R(a_{j+1})R(a_{j+2})x|y > = < (xa_{j+2})a_{j+1}|y >$$

$=< xa_{j+2}|a_{j+1}y >=< x|a_{j+2}(a_{j+1}y) >=< x|L(a_{j+2})L(a_{j+1})y) =< x|\theta_j(a)y >,$

which is Eq.(2.30e). Then, setting $y \to \sigma_j(a)y$ in Eq.(2.30e) and noting Eq.(2.30c), we obtain Eq.(2.30f).

We next rewrite Eqs.(2.29) as

$$\sigma_j(a)(x) = 2 < a_{j+1}|x > a_{j+2} - a_j x \tag{2.32a}$$

$$\theta_j(a)x = 2 < a_{j+2}|x > a_{j+1} - xa_j \tag{2.32b}$$

since we have

$$\sigma_j(a)x = (xa_{j+2})a_{j+1} =$$

$$2 < x|a_{j+1} > a_{j+2} - (a_{j+1}a_{j+2})x = 2 < a_{j+1}|x > a_{j+2} - a_j x$$

and

$$\theta_j(a)x = a_{j+2}(a_{j+1}x) =$$

$$2 < a_{j+2}|x > a_{j+1} - x(a_{j+1}a_{j+2}) = 2 < a_{j+2}|x > a_{j+1} - xa_j$$

in view of Eqs.(2.3) and (2.28). We then calculate

$$\sigma_{j+2}(a)\sigma_{j+1}(a)x = 2 < a_j|\sigma_{j+1}(a)x > a_{j+1} - a_{j+2}(\sigma_{j+1}(a)x)$$

$$= 2 < a_j|2 < a_{j+2}|x > a_j - a_{j+1}x > a_{j+1} - a_{j+2}\{2 < a_{j+2}|x > a_j - a_{j+1}x\}$$

$$= \{4 < a_{j+2}|x >< a_j|a_j > -2 < a_j a_{j+1}|x >\}a_{j+1}$$

$$-2 < a_{j+2}|x > a_{j+2}a_j + a_{j+2}(a_{j+1}x)$$

$$= \{4 < a_{j+2}|x > -2 < a_{j+2}|x >\}a_{j+1}$$

$$-2 < a_{j+2}|x > a_{j+1} + a_{j+2}(a_{j+1}x) = a_{j+2}(a_{j+1}x)$$

$$= \theta_j(a)x,$$

and so $\sigma_{j+2}(a)\sigma_{j+1}(a) = \theta_j(a)$. Multiplying $\sigma_j(a)$ from the left and noting Eq.(2.30c), we obtain $\sigma_{j+2}(a)\sigma_{j+1}(a)\sigma_j(a) = Id$. If we further multiply this by $\theta_j(a)\theta_{j+1}(a)\theta_{j+2}(a)$ from the left, it also reproduces

$$\theta_j(a)\theta_{j+1}(a)\theta_{j+2}(a) = Id,$$

proving Eq.(2.30d).

It now remains to prove Eqs.(2.30a) and (2.30b). We calculate

$$(\sigma_j(a)x)(\sigma_{j+1}(a)y) = (\sigma_j(a)x)\{(ya_j)a_{j+2}\} =$$

$$2 < \sigma_j(a)x|a_{j+2} > ya_j - a_{j+2}\{(ya_j)(\sigma_j(a)x)\}$$

$$= 2 < x|\theta_j(a)a_{j+2} > ya_j - a_{j+2}\{(ya_j)(2 < x|a_{j+1} > a_{j+2} - a_jx)\}$$

$$= 2 < x|a_{j+2}(a_{j+1}a_{j+2}) > ya_j - 2 < x|a_{j+1} > a_{j+2}\{(ya_j)a_{j+2}\}$$

$$+a_{j+2}\{(ya_j)(a_jx)\}$$

$$= 2 < x| < a_{j+2}|a_{j+2} > a_{j+1} > ya_j - 2 < x|a_{j+1} > ya_j$$

$$+a_{j+2}\{2 < y|a_jx > a_j - < a_j|a_j > xy\}$$

$$= 2 < xy|a_j > a_{j+1} - a_{j+2}(xy) = \sigma_{j+2}(a)(xy),$$

where we utilized Eq.(2.4). This proves Eq.(2.30a). If we replace x and y by $\theta_{j+1}(a)x$ and $\theta_{j+2}(a)y$, respectively, in Eq.(2.30a), we also obtain Eq.(2.30b). This completes the proof.□

Noting

$$\sigma(a) = (\sigma_1(a), \sigma_2(a), \sigma_3(a)) \in Trig(A) \tag{2.33a}$$

$$\theta(a) = (\theta_1(a), \theta_2(a), \theta_3(a)) \in Trig(A) \tag{2.33b}$$

by Theorem 2.5, these generate a subgroup of $Trig(A)$. However, except for the cases of $Dim\ A = 1$ and $Dim\ A = 2$, it is not an invariant subgroup of $Trig(A)$ for the following reason. For any $g = (g_1, g_2, g_3) \in Trig(A)$, Eqs.(1.25) yield

$$g_jL(x)L(y) = L(g_{j+1}x)g_{j+2}L(y) = L(g_{j+1}x)L(g_jy)g_{j+1}, \tag{2.34a}$$

$$g_jR(x)R(y) = R(g_{j+2}x)g_{j+1}R(y) = R(g_{j+2}x)R(g_jy)g_{j+2}, \tag{2.34b}$$

implying that

$$g_j\theta_k(a) = L(g_{j+1}a_{k+2})L(g_ja_{k+1})g_{j+1} \tag{2.34c}$$

$$g_j\sigma_k(a) = R(g_{j+2}a_{k+1})R(g_ja_{k+2})g_{j+2}. \tag{2.34d}$$

We next note

$$(g_ja_j)(g_{j+1}a_{j+1}) = g_{j+2}(a_ja_{j+1}) = g_{j+2}a_{j+2};$$

thus, we have

$$ga = (g_1a_1, g_2a_2, g_3a_3) \in \Sigma. \tag{2.35}$$

Moreover, we similarly have

$$\phi g = (g_2, g_3, g_1) \in Trig(A)$$

$$\phi^2 g = (g_3, g_1, g_2) \in Trig(A)$$

for $\phi : 1 \to 2 \to 3 \to 1$. Then, Eqs.(2.34) are rewritten as

$$g_j\theta_k(a)g_{j+1}^{-1} = \theta_k((\phi^{(j-k-1)}g)a) \tag{2.36a}$$

$$g_j\sigma_k(a)g_{j+2}^{-1} = \sigma_k((\phi^{(j-k+1)}g)a) \tag{2.36b}$$

for any $j, k = 1, 2, 3$. In particular, for $j = k$, we obtain

$$g_j\theta_j(a)g_{j+1}^{-1} = \theta_j((\phi g)a) \tag{2.37a}$$

$$g_j\sigma_j(a)g_{j+2}^{-1} = \sigma_j((\phi^2 g)a). \tag{2.37b}$$

Since the left sides of Eqs.(2.37) are not of the form $g_j \times g_j$, the subgroup G_0 generated by $\sigma_j(a)$ and $\theta_j(a)$ is not generally invariant under $Trig(A)$. However, the cases of $Dim\ A = 1$ and $Dim\ A = 2$ are exceptional since they are Abelian and hence satisfy $R(x) = L(x)$. Then, we can rewrite

$$\sigma_j(a) = R(a_{j+1})R(a_{j+2}) = R(a_{j+1})L(a_{j+2})$$

$$\theta_j(a) = L(a_{j+2})L(a_{j+1}) = L(a_{j+2})R(a_{j+1}).$$

Then, Eqs.(1.26) show that these are invariant under $Trig(A)$.

Returning to the original problem, we can construct some invariant subgroups of $Trig(A)$. Let G be the subgroup generated by

$$\sigma_j(a)\theta_j(b) \quad and \quad \theta_j(a)\sigma_j(b) \tag{2.38a}$$

or

$$\sigma_j(a)\sigma_j(b)\sigma_j(c) \quad and \quad \theta_j(a)\theta_j(b)\theta_j(c) \tag{2.38b}$$

for any $a, b, c \in \Sigma$. Then, by Eq.(2.37), $X = (X_1, X_2, X_3) \in G$ will be transformed by $Trig(A)$ into

$$g_j X_j g_j^{-1} = X_j^{'}$$

for another $X^{'} = (X_1^{'}, X_2^{'}, X_3^{'}) \in G$, and G is an invariant subgroup of $Trig(A)$.

In the final part of this section, for the 8-dimensional pseudo-octonion algebra, we note that there is an example of elements in Σ as follows.

On the basis of the notations and Eqs.(4.26) in ([O.95]), we have a basis $e_1, \cdots, e_8$ such that

$$e_j e_k = \sum_{l=1}^{8} (\sqrt{3} d_{jkl} \mp f_{jkl}) e_l \quad and \quad < e_i | e_j >= \delta_{ij}.$$

For the special basis e_1, e_2, e_3, and the case of the $+$ sign in $\mp$, we obtain

$$e_1 e_2 = e_3, \quad e_2 e_3 = e_1, \quad e_3 e_1 = e_2,$$

by using the values $d_{12l} = d_{23l} = d_{31l} = 0$ $(l = 1, \cdots, 8)$, and $f_{123} = f_{231} = f_{312} = 1$, $f_{12l} = 0$ $(if\ l \neq 3)$, $f_{23l} = 0$ $(if\ l \neq 1)$, and $f_{31l} = 0$ $(if\ l \neq 2)$ (for the explicit numerical values of d_{jkl}, f_{jkl} and an application to physics, see [G.62]).

Hence, this implies that there is an element $(e_1, e_2, e_3) \in \Sigma$ satisfying the assumption in Theorem 2.5. Furthermore, as this algebra is defined by 3×3 matrices, it is easy to see that the transpose of a matrix induces an involution with respect to $\overline{x} = {}^t x$.

3 Local Triality Relations

In this section, we will study the relationship between the invariant subgroup of $Trig(A)$ given in section 2 and its local correspondent $s \circ Lrt(A)$ defined in Eq.(1.18).

In section 3 and section 4, we assume that the algebra A is a symmetric composition algebra over a field F of characteristic not 2 unless otherwise specified.

Let

$$\Sigma = \{a = (a_1, a_2, a_3) \in A^3 | a_j a_{j+1} = a_{j+2}, < a_j | a_j >= 1, \ \forall j = 1, 2, 3\} \tag{3.1}$$

as in Eq.(2.28). For a given $a = (a_1, a_2, a_3) \in \Sigma$, we introduce another triple $\Lambda(a)$ as

$$\Lambda(a) = \{p = (p_1, p_2, p_3) \in A^3 |$$

$$a_j p_{j+1} + p_j a_{j+1} = p_{j+2}, < p_j | a_j >= 0, \forall j = 1, 2, 3\}, \tag{3.2}$$

where the indices over j are defined modulo 3 with $p_{j+3} = p_j$. We note that $\Lambda(a) = \text{span} < p_1, p_2 >$ for p_1, p_2 satisfying $< p_j | a_j >= 0$, only for $j = 1, 2$, as we see below. For any such p_1 and p_2, we define p_3 by

$$p_3 = a_1 p_2 + p_1 a_2,$$

and we then calculate

$$< a_3 | p_3 >=< a_3 | a_1 p_2 > + < a_3 | p_1 a_2 >=< a_3 a_1 | p_2 > + < a_2 a_3 | p_1 >$$

$$=< a_2 | p_2 > + < a_1 | p_1 >= 0$$

$$a_2 p_3 + p_2 a_3 = a_2(a_1 p_2 + p_1 a_2) + p_2(a_1 a_2) =$$

$$2 < a_2 | p_2 > a_1 + < a_2 | a_2 > p_1 = p_1,$$

and

$$a_3 p_1 + p_3 a_1 = (a_1 a_2) p_1 + (a_1 p_2 + p_1 a_2) a_1$$

$$= (a_1 a_2) p_1 + (p_1 a_2) a_1 + (a_1 p_2) a_1 = 2 < a_1 | p_1 > a_2 + < a_1 | a_1 > p_2 = p_2,$$

implying that $(p_1, p_2, p_3) \in A(a)$. Note that $\Lambda(a)$ is then a vector space over the field F, in contrast to $\sum$.

Moreover, we further define $q_j \in A$ by

$$q_j = a_{j+1} p_{j+2} = p_j - p_{j+1} a_{j+2}. \tag{3.3}$$

We then find

Lemma 3.1

Under the above assumption, we have $q = (q_1, q_2, q_3) \in \Lambda(p)$. *Conversely,* p_j *is computed by*

$$p_j = q_{j+1}a_{j+2} = q_j - a_{j+1}q_{j+2}. \tag{3.4}$$

Proof

We first note

$$< q_j|a_j >=< a_{j+1}p_{j+2}|a_{j+1}a_{j+2} >=< a_{j+1}|a_{j+1} >< p_{j+2}|a_{j+2} >= 0.$$

Also, we calculate

$$q_{j+1}a_{j+2} = (a_{j+2}p_{j+3})a_{j+2} =< a_{j+2}|a_{j+2} > p_{j+3} = p_j.$$

Finally, we find

$$a_jq_{j+1} + q_ja_{j+1} = a_j(a_{j+2}p_{j+3}) + (a_{j+1}p_{j+2})a_{j+1}$$

$$= 2 < a_j|p_j > a_{j+2} - p_j(a_{j+2}a_j) + < a_{j+1}|a_{j+1} > p_{j+2}$$

$$= -p_ja_{j+1} + p_{j+2} = q_{j+2}.$$

This proves Lemma 3.1.□

Theorem 3.2

For any $a \in \Sigma$ *and* $p \in \Lambda(a)$, *if we introduce* $D_j(a,p) \in End(A)$ *as*

$$D_j(a,p)x = (p_{j+1}x)a_{j+1} + a_j(xq_j), \tag{3.5}$$

then,

$$D_j(a,p)(xy) = (D_{j+1}(a,p)x)y + x(D_{j+2}(a,p)y) \tag{3.6}$$

is satisfied, implying that

$$D(a,p) = (D_1(a,p), D_2(a,p), D_3(a,p)) \in s \circ Lrt(A).$$

For a proof of this theorem, we require the following.

Lemma 3.3

Under the assumption in Theorem 3.2, we have

$$D_j(a,p)x = 2 < q_{j+2}|x > a_{j+2} - 2 < a_{j+2}|x > q_{j+2} + a_j(xp_j) \qquad (3.7a)$$

$$= 2 < p_{j+2}|x > a_{j+2} - 2 < a_{j+2}|x > p_{j+2} + (q_{j+1}x)a_{j+1}. \qquad (3.7b)$$

Proof

We calculate

$$a_j(xq_j) = a_j\{xp_j - x(p_{j+1}a_{j+2})\}$$

$$= a_j(xp_j) - a_j\{2 < x|a_{j+2} > p_{j+1} - a_{j+2}(p_{j+1}x)\}$$

$$= a_j(xp_j) - 2 < x|a_{j+2} > a_jp_{j+1} + a_j\{a_{j+2}(p_{j+1}x)\}$$

$$= a_j(xp_j) - 2 < x|a_{j+2} > a_jp_{j+1} + 2 < a_j|p_{j+1}x > a_{j+2} - (p_{j+1}x)(a_{j+2}a_j)$$

$$= a_j(xp_j) - 2 < x|a_{j+2} > q_{j+2} + 2 < a_jp_{j+1}|x > a_{j+2} - (p_{j+1}x)a_{j+1}$$

$$= a_j(xp_j) - 2 < x|a_{j+2} > q_{j+2} + 2 < q_{j+2}|x > a_{j+2} - (p_{j+1}x)a_{j+1},$$

from which we obtain

$$D_j(a,p)x = a_j(xq_j) + (p_{j+1}x)a_{j+1}$$

$$= a_j(xp_j) - 2 < x|a_{j+2} > q_{j+2} + 2 < q_{j+2}|x > a_{j+2},$$

which is Eq.(3.7a). To show Eq.(3.7b), we note

$$a_j(xp_j) = a_j\{x(q_{j+1}a_{j+2})\} = a_j\{2 < x|a_{j+2} > q_{j+1} - a_{j+2}(q_{j+1}x)\}$$

$$= 2 < x|a_{j+2} > a_jq_{j+1} - a_j\{a_{j+2}(q_{j+1}x)\}. \qquad (3.8)$$

However, using $q_{j+1} = a_{j+2}p_j$, we further compute

$$a_jq_{j+1} = a_j(a_{j+2}p_j) = 2 < a_j|p_j > a_{j+2} - p_j(a_{j+2}a_j) = -p_ja_{j+1}$$

and

$$a_j\{a_{j+2}(q_{j+1}x)\} = 2 < a_j|q_{j+1}x > a_{j+2} - (q_{j+1}x)(a_{j+2}a_j)$$

$$= 2 < a_jq_{j+1}|x > a_{j+2} - (q_{j+1}x)a_{j+1}$$

$$= -2 < p_j a_{j+1} | x > a_{j+2} - (q_{j+1}x)a_{j+1};$$

thus, Eq.(3.8) becomes

$$D_j(a,p)x = 2 < q_{j+2}|x > a_{j+2} - 2 < a_{j+2}|x > q_{j+2}$$

$$-2 < x|a_{j+2} > p_j a_{j+1} + 2 < p_j a_{j+1}|x > a_{j+2} + (q_{j+1}x)a_{j+1} =$$

$$2 < (q_{j+2} + p_j a_{j+1})|x > a_{j+2} - 2 < a_{j+2}|x > (q_{j+2} + p_j a_{j+1}) +$$

$$(q_{j+1}x)a_{j+1}$$

$$= 2 < p_{j+2}|x > a_{j+2} - 2 < a_{j+2}|x > p_{j+2} + (q_{j+1}x)a_{j+1},$$

which is Eq.(3.7b). This completes the proof of Lemma 3.3.□

We are now in a position to prove Theorem 3.2. We calculate

$$(D_j(a,p)x)y + x(D_{j+1}(a,p)y)$$

$$= \{2 < q_{j+2}|x > a_{j+2} - 2 < a_{j+2}|x > q_{j+2} + a_j(xp_j)\}y$$

$$+x\{2 < p_{j+3}|y > a_{j+3} - 2 < a_{j+3}|y > p_{j+3} + (q_{j+2}y)a_{j+2}\}$$

$$= 2 < q_{j+2}|x > a_{j+2}y - 2 < a_{j+2}|x > q_{j+2}y + \{a_j(xp_j)\}y$$

$$+2 < p_j|y > xa_j - 2 < a_j|y > xp_j + x\{(q_{j+2}y)a_{j+2}\}.$$

However, we note

$$\{a_j(xp_j\}y = 2 < a_j|y > xp_j - \{y(xp_j)\}a,$$

$$= 2 < a_j|y > xp_j - \{2 < y|p_j > x - p_j(xy)\}a_j$$

$$= 2 < a_j|y > xp_j - 2 < y|p_j > xa_j + \{p_j(xy)\}a_j$$

and

$$x\{(q_{j+2}y)a_{j+2}\} = 2 < x|a_{j+2} > q_{j+2}y - a_{j+2}\{(q_{j+2}y)x\}$$

$$= 2 < x|a_{j+2} > q_{j+2}y - a_{j+2}\{2 < q_{j+2}|x > y - (xy)q_{j+2}\}$$

$$= 2 < x|a_{j+2} > q_{j+2}y - 2 < q_{j+2}|x > a_{j+2}y + a_{j+2}\{(xy)q_{j+2}\},$$

and thus, we obtain

$$\{D_j(a,p)x\}y + x\{D_{j+1}(a,p)y\}$$

$$= 2 < q_{j+2}|x > a_{j+2}y - 2 < a_{j+2}|x > q_{j+2}y + 2 < a_j|y > xp_j -$$
$$2 < y|p_j > xa_j + \{p_j(xy)\}a_j + 2 < p_j|y > xa_j - 2 < a_j|y > xp_j$$
$$+2 < x|a_{j+2} > q_{j+2}y - 2 < q_{j+2}|x > a_{j+2}y + a_{j+2}\{(xy)q_{j+2}\}$$
$$= \{p_j(xy)\}a_j + a_{j+2}\{(xy)q_{j+2}\} = D_{j+2}(a,p)(xy).$$

Letting $j \to j+1$, this proves Theorem 3.2.□

For an example of $\Lambda(a)$ in the case of 8-dimensional pseudo-octonion algebra, if $a = (e_1, e_2, e_3) \in \Sigma$, as in the end of section 2, then we note

$$p = (e_8, e_8, e_1 + e_2) \in \Lambda(a)$$

since $e_1e_8 + e_8e_2 = e_1 + e_2$ and $< e_8|e_1 * e_2 >$, etc.

Next, for another proof of Theorem 3.2, we will show that $D_j(a,p)$ may be regarded as a local sum of some elements of $\sigma_j(x)\theta_j(y)$ that generate the invariant subgroup of $\mathrm{Trig}(A)$, as we noted in the previous section. For this purpose, we temporarily assume that the field F is either real or complex with a formal infinitesimal variable $\varepsilon \in F$ with $|\varepsilon| << 1$. We heuristically proceed as follows.

Let $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3) \in \Sigma$ and suppose that $b \in \Sigma$ is infinitesimally close to $a \in \Sigma$. Then we write

$$b_j = a_j + \varepsilon p_j + O(\varepsilon^2) \quad (j = 1, 2, 3) \tag{3.9}$$

for some $p_j \in A$. Thus, the condition $b_jb_{j+1} = b_{j+2}$ with $< b_j|b_j >= 1$ is equivalent to $p = (p_1, p_2, p_3) \in \Lambda(a)$ with $< a_j|p_j >= 0$, and we calculate

$$\theta_j(b)x = \theta_j(a)x + \varepsilon\{2 < p_{j+2}|x > a_{j+1} + 2 < a_{j+2}|x > p_{j+1} - xp_j\} + O(\varepsilon^2),$$

allowing us to write

$$\sigma_j(a)\theta_j(b)x = x + \varepsilon Dx + O(\varepsilon^2) \tag{3.10a}$$

for some $D \in \mathrm{End}(A)$ defined by

$$Dx = 2 < p_{j+2}|x > \sigma_j(a)a_{j+1} + 2 < a_{j+2}|x > \sigma_j(a)p_{j+1} - \sigma_j(a)(xp_j). \tag{3.10b}$$

We calculate

$$\sigma_j(a)a_{j+1} = (a_{j+1}a_{j+2})a_{j+1} = a_j a_{j+1} = a_{j+2}$$

$$\sigma_j(a)p_{j+1} = (p_{j+1}a_{j+2})a_{j+1} = (p_j - a_{j+1}p_{j+2})a_{j+1}$$

$$= p_j a_{j+1} - < a_{j+1}|a_{j+1} > p_{j+2} = p_j a_{j+1} - p_{j+2} = -a_j p_{j+1},$$

which implies that

$$Dx = 2 < p_{j+2}|x > a_{j+2} - 2 < a_{j+2}|x > a_j p_{j+1}$$

$$-\{2 < a_{j+1}|xp_j > a_{j+2} - a_j(xp_j)\}$$

$$= 2 < (p_{j+2} - p_j a_{j+1})|x > a_{j+2} - 2 < a_{j+2}|x > a_j p_{j+1} + a_j(xp_j)$$

$$= 2 < a_j p_{j+1}|x > a_{j+2} - 2 < a_{j+2}|x > a_j p_{j+1} + a_j(xp_j) = D_j(a,p)x.$$

Therefore, Eqs.(2.20) are rewritten as

$$\sigma_j(a)\theta_j(b) = 1 + \varepsilon D_j(a,p) + O(\varepsilon^2) \tag{3.11}$$

for $D_j(a,p)$ defined by Eq.(3.5) or (3.7).

Then, since $\sigma(a)\theta(b) \in Trig(A)$, we have

$$\sigma_j(a)\theta_j(b)(xy) = (\sigma_{j+1}(a)\theta_{j+1}(b)x)y + x(\sigma_{j+2}(a)\theta_{j+2}(b)y),$$

which implies

$$\{1 + \varepsilon D_j(a,p) + O(\varepsilon^2)\}(xy) =$$

$$(\{1 + \varepsilon D_{j+1}(a,p) + O(\varepsilon^2)\}x)y + x\{1 + \varepsilon D_{j+2}(a,b) + O(\varepsilon^2)\}y.$$

Hence, from the above relation, we obtain

$$D_j(a,p)(xy) = (D_{j+1}(a,p)x)y + x(D_{j+2}(a,p)y),$$

giving an alternative heuristic approach to the proof of Theorem 3.2.

We note that all the relations given so far are invariant under the action of ϕ defined by

$$\phi : a_j \to a_{j+1},\ p_j \to p_{j+1},\ q_j \to q_{j+1}, \tag{3.12}$$

which leads to

$$D_j(a,p) \to D_{j+1}(a,p).$$

Next, let us consider the relation between $D_j(a,p)$ and the normal triality map $d_j(x,y)$ defined by

$$d_1(x,y) = R(y)L(x) - R(x)L(y) \tag{3.13a}$$

$$d_2(x,y) = L(y)R(x) - L(x)R(y), \tag{3.13b}$$

where $d_3(x,y)$ for the symmetric composition algebra is given by ([O.05])

$$d_3(x,y)z = 4\{< x|z > y - < y|z > x\}. \tag{3.13c}$$

If we choose $p_3 = 0$, this leads to $q_1 = 0$, $q_2 = p_2 = a_3 p_1$, and $q_3 = a_1 p_2 = -p_1 a_2$ from Eqs.(3.2),(3.3), and (3.4). For any two arbitrary constants $\alpha, \beta \in F$ with $\beta \neq 0$, we then set

$$u = \frac{1}{2\beta}(p_2 + \alpha a_2), \ v = \beta a_2. \tag{3.14}$$

Then, we can readily verify the validity of

$$D_j(a,p) = d_j(u,v). \tag{3.15}$$

Similarly, if we choose $p_1 = 0$ or $p_2 = 0$ and note Eq.(3.12), this also leads to another identity similar to Eq.(3.15). Then, the original $D_j(a,p)$ can be expressed as linear combinations of these $d_j(u,v)$ for $j = 1,2,3$.

For the symmetric composition algebra, $d_j(x,y)$ given by Eqs.(3.13) satisfy the cubic equation

$$(d_j(x,y))^3 = \Delta(x,y)d_j(x,y), \ (j = 1,2,3) \tag{3.16a}$$

$$\Delta(x,y) = 4(< x|y >< x|y > - < x|x >< y|y >), \tag{3.16b}$$

as we may readily verify. In fact, for the special case of $j = 1,2$, we have the stronger relation

$$(d_j(x,y))^2 = \Delta(x,y)Id, \ (j = 1,2). \tag{3.17}$$

Then, its exponential map can be computed to be

$$\exp\{\lambda d_j(x,y)\} =$$

$$1 + \frac{sinh(\lambda\sqrt{\Delta(x,y)}}{\sqrt{\Delta(x,y)}} d_j(x,y) + \frac{1}{\Delta(x,y)}\{cosh(\lambda\sqrt{\Delta(x,y)}) - 1\}(d_j(x,y)^2. \tag{3.18}$$

However, it is more difficult to evaluate $\exp\{\sum_{j=1}^{3} \lambda_j D_j(a,p)\}$, $\lambda_j \in F$ by a similar method.

4 Automorphism Group of Symmetric Composition Algebras

In this section, we will discuss the automorphisms of the symmetric composition algebras, particularly for the Cayley algebra.

Let $a \in A$ be a nontrivial idempotent of a symmetric composition algebra A; thus,

$$aa = a, \ < a|a >= 1. \tag{4.1}$$

Then, the triple (a,a,a) is an element of $\sum$ in (2.28) with $\sigma_1(a) = \sigma_2(a) = \sigma_3(a)$ and $\theta_1(a) = \theta_2(a) = \theta_3(a)$. In this case, Theorem 2.5 becomes

Theorem 4.1

Let $a \in A$ be a nontrivial idempotent of A satisfying Eq.(4.1). If we set

$$\sigma(a) = R(a)R(a) \tag{4.2a}$$

$$\theta(a) = L(a)L(a), \tag{4.2b}$$

then they satisfy

(1)

$$\sigma(a)(xy) = (\sigma(a)x)(\sigma(a)y) \tag{4.3a}$$

(2)

$$\theta(a)(xy) = (\theta(a)x)(\theta(a)y) \tag{4.3b}$$

(3)

$$\sigma(a)\theta(a) = \theta(a)\sigma(a) = Id \tag{4.3c}$$

(4)

$$(\sigma(a))^3 = (\theta(a))^3 = Id \tag{4.3d}$$

(5)

$$(\sigma(a))^3 = (\theta(a))^3 = Id \tag{4.3e}$$

(6)

$$< \sigma(a)x|\sigma(a)y >=< \theta(a)x|\theta(a)y >=< x|y > . \tag{4.3f}$$

Therefore, both $\sigma(a)$ and $\theta(a)$ are automorphisms of A of order 3.

Moreover, by restricting $g_1 = g_2 = g_3(\equiv g) \in Aut(A)$ in Eqs.(2.37), they generate an invariant subgroup of $Aut(A)$.

In particular, from this theorem, the case of a para-Hurwitz algebra with the para-unit e is of interest, which can be obtained from the Hurwitz algebra A^*, i.e., a unital composition algebra with the bilinear product $x * y$ satisfying

$$xy = \overline{x * y} = \bar{y} * \bar{x}, \tag{4.4}$$

with $e * x = x * e = x$, and the involution map $x \to \bar{x}$ given by [S.66]

$$\bar{x} = 2 < e|x > e - x. \tag{4.5}$$

Introducing multiplication operators $l(x)$ and $r(x) \in \mathrm{End}(A^*)$ as

$$l(x)y = x * y, \ r(x)y = y * x, \tag{4.6}$$

we find

Corollary 4.2

Let A^* be a Hurwitz algebra over a field F of characteristic $\neq 2$ with the unit element e and the bilinear product $x * y(= \overline{xy})$. For any $a \in A^*$ satisfying

$$< a|a >= 1, \text{ and } 2 < e|a >= -1, \tag{4.7}$$

we set

$$\sigma(a) = l(\bar{a})r(a) = r(a)l(\bar{a}). \tag{4.8}$$

We then have

(i)

$$\overline{\sigma(a)}(x * y) = (\sigma(a)x) * (\sigma(a)y) \tag{4.9a}$$

(ii)

$$\sigma(a)\sigma(\bar{a}) = Id \tag{4.9b}$$

(iii)

$$(\sigma(a))^3 = Id \tag{4.9c}$$

(iv)

$$< \sigma(a)x|\sigma(a)y >=< x|y > \tag{4.9d}$$

for any $x, y \in A^*$. *In particular,* $\sigma(a) \in Aut(A^*)$.

Proof

This is simply a special case of Theorem 4.1 for the following reason. We first note that any $a \in A^*$ satisfying Eq.(4.7) with $\bar{a} = 2 < a|e > e - a$ satisfies $a * a = \bar{a}$, and hence $aa = a$ since any $x \in A^*$ obeys the quadratic relation $x * \bar{x} =< x|x > e$, or equivalently ([S.66])

$$x * x = 2 < e|x > x - < x|x > e, \tag{4.10}$$

and $\sigma(a) = R(a)R(a)$ in A. Then, Eq.(4.3a) is rewritten as

$$\overline{\sigma(a)}(x * y) = (\sigma(a)x) * (\sigma(a)y). \tag{4.11}$$

Setting $y = e$, this yields

$$\overline{\sigma(a)}x = (\sigma(a)x) * (\sigma(a)e).$$

However,

$$\sigma(a)e = l(\bar{a})r(a)e = \bar{a} * (e * a) = \bar{a} * a =< a|a > e = e,$$

implying that $\overline{\sigma(a)}x = (\sigma(a)x) * e = \sigma(a)x$ and then $\overline{\sigma(a)} = \sigma(a)$.

Then, Eq.(4.11) leads to Eq.(4.9a), from which Eq.(4.8) gives $\theta(a) = \sigma(\bar{a})$ and hence Eq.(4.9b). This completes the proof.□

Remark 4.3

Let G be the group generated by $\sigma(a)$. Then, G is an invariant subgroup of $Auto(A^*)$, as we can see by setting $g_1 = g_2 = g_3(\equiv g) \in Aut(A^*)$ in Eqs.(2.36). Suppose now that A^* is the Cayley algebra with basis $e_0(\equiv e), e_1, \cdots, e_7$ satisfying

$$< e_\mu | e_\nu >= \delta_{\mu\nu}. \ (\mu, \nu = 0, 1, 2, \cdots, 7).$$

Then, expressing

$$a = \frac{1}{2}(-e + \sum_{\lambda=1}^{7} \alpha_\lambda e_\lambda)$$

for $\alpha_\lambda \in F$, the condition Eq.(4.7) is equivalent to the validity of

$$\sum_{\lambda=1}^{7} (\alpha_\lambda)^2 = 3.$$

This implies that there is an idempotent element a in the assumption of Theorem 4.1 and Corollary 4.2 for the Cayley algebra since $aa = a$ and $< a|a >= 1$. Furthermore, if the field F is of characteristic not 3, we can identify such an element as a point on the 6-sphere S^6. Hence, we may find idempotent elements for other algebras by the same method. Moreover, for any $b \in A^*$ satisfying $< b|b >= 1$ and $2 < b|e >= -1$, $c = \sigma(b)a$ preserves the same condition since

$$< c|c >=< \sigma(b)a|\sigma(b)a >=< a|a >= 1,$$

$$2 < c|e >= 2 < \sigma(b)a|e >= 2 < a|\sigma(\bar{b})e >= 2 < a|e >= -1.$$

Therefore, $\sigma(b)$ acts upon S^6, although it may not be transitive. However, for any $b, c \in S^6$, we can find $a \in S^6$ satisfying

$$\sigma(a)b = c,$$

provided that $2 < b|c > +1 \neq 0$ with the field F being quadratically closed. The solution is given by

$$a = \frac{1}{1 + \lambda + \lambda^2}\{\lambda(b + c) - \lambda^2 e - c * b\} \in S^6,$$

where $\lambda \in F$ is a solution of the quadratic equation

$$1 + \lambda + \lambda^2 = 2 < b|c > +1 \ (\neq 0).$$

If b and c satisfy $2 < b|c > +1 = 0$, then we cannot find such $a \in S^6$. However, in that case, we choose $b' \in S^6$ satisfying $2 < c|b' > +1 \neq 0$ and $2 < b|b' > +1 \neq 0$; thus, there exist $a_1, a_2 \in S^6$ satisfying

$$\sigma(a_1)b = b' \text{ and } \sigma(a_2)b' = c,$$

which leads to $\sigma(a_2)\sigma(a_1)b = c$. This implies that the group G now acts transitively upon S^6.

We also note that G is an invariant subgroup of $Aut(A^*)$; we have $t\sigma(a)t^{-1} = \sigma(ta)$ for any $t \in Aut(A^*)$ if we choose $t = \sigma(b)$. This gives

$$\sigma(b)\sigma(a) = \sigma(\sigma(b)a)\sigma(b).$$

The corresponding derivation algebra is obtained from Eq.(3.5) by setting $a_1 = a_2 = a_3(\equiv a)$, $p_1 = p_2 = p_3(\equiv p)$ and $q_1 = q_2 = q_3(\equiv q)$. Then,

$$D(a,p) := D_1(a,p) = D_2(a,p) = D_3(a,p)$$

is given by

$$D(a,p)x = (px)a + a(xq) = \bar{a} * (p * x) + (x * q) * \bar{a} \qquad (4.12a)$$

or

$$D(a,p) = l(\bar{a})l(p) + r(\bar{a})r(q). \qquad (4.12b)$$

Note that there are many constrained relations among a, p, and q as we see in the following.

Lemma 4.4

Under the assumption in Corollary 4.2, we have
(1)

$$< a|a >= 1, \ < p|p >=< q|q >= 2 < p|q >, \ < p|a >=< q|a >= 0 \qquad (4.13a)$$

(2)

$$2 < a|e >= -1,\ < p|e >=< q|e >= 0,\ \bar{a} = -e - a \tag{4.13b}$$

(3)

$$q = -p * \bar{a} = -a * p,\ p = -\bar{a} * q = -q * a \tag{4.13c}$$

(4)

$$q * p =< p|p > a,\ p * q =< p|p > \bar{a}. \tag{4.13d}$$

Proof

From Eq.(3.2), we calculate

$$p = ap + pa = \bar{p} * \bar{a} + \bar{a} * \bar{p} = 2 < \bar{p}|e > \bar{a} + 2 < \bar{a}|e > \bar{p} - 2 < \bar{a}|\bar{p} > e$$

$$= 2 < p|e > \bar{a} - \bar{p} - 0 = 2 < p|e > \bar{a} - (2 < p|e > e - p)$$

$$= 2 < p|e > (\bar{a} - e) + p,$$

which gives $< p|e >= 0$. Then, Eq.(3.3) leads to

$$q = ap = \bar{p} * \bar{a} = -p * \bar{a}.$$

Also, we calculate

$$< q|e >=< ap|e >=< a|pe >=< a|\bar{p} >= - < a|p >= 0,$$

and hence $< q|e >= 0$. Now, Eq.(3.4) is rewritten as

$$p = qa = \bar{a} * \bar{q} = -\bar{a} * p.$$

Moreover,

$$p * q = -p * \bar{q} = (\bar{a} * q) * \bar{q} = \bar{a} < q|q > e =< q|q > \bar{a}$$

and

$$< q|p >=< q|-\bar{a}*q >= - < e*q|\bar{a}*q >= - < e|\bar{a} >< q|q >= \frac{1}{2} < q|q >,$$

$$< p|a >=< -q * a|a >= - < q * a|e * a >= - < q|e >< a|a >= 0,$$

$$< q|a >= - < a * p|a * e >= - < a|a >< p|e >= 0.$$

Finally, we obtain

$$p * q + q * p = 2 < p|e > q + 2 < q|e > p - 2 < p|q > e = - < p|p > e,$$

completing the proof.□

Lemma 4.5

Under the assumption in Corollary 4.2, for any $f, g, x \in A^*$ and unit element e, we have

(1)

$$f * (g * x) = x * (f * g) - 2 < f|e > (x * g) + 2 < g|e > (f * x) + 2 < f|x > g - 2 < g|x > f \tag{4.14a}$$

(2)

$$(x * f) * g = (f * g) * x + 2 < f|e > (x * g) - 2 < g|e > (f * x) - 2 < f|x > g + 2 < g|x > f \tag{4.14b}$$

(3)

$$f*(x*g) = -x*(f*g)+2 < x|e > (f*g)+2 < f|e > (x*g)-2 < f|x > g \tag{4.14c}$$

(4)

$$(f*x)*g = -(f*g)*z+2 < x|e > (f*g)+2 < g|e > (f*x)-2 < g|x > f. \tag{4.14d}$$

Proof

We calculate

$$f * (x * g) + x * (f * g) = \{2 < f|e > x + 2 < x|e > f - 2 < f|x > e\} * g$$

$$= 2 < f|e > x * g + 2 < x|e > f * g - 2 < f|x > g$$

and

$$(f*x)*g+(f*g)*x=f*\{2<g|e>x+2<x|e>g-2<x|g>e\}$$
$$=2<g|e>g*x+2<x|e>f*g-2<x|g>f,$$

which give Eqs.(4.14c) and (4.14d). We then obtain

$$f*(g*x)+f*(x*g)=f*\{2<x|e>g+2<g|e>x-2<x|g>e\}$$
$$=2<x|e>f*g+2<g|e>f*x-2<x|g>f$$

and

$$(f*x)*g+(x*f)*g=\{2<x|e>f+2<f|e>x-2<f|x>e\}*g$$
$$=2<x|e>f*g+2<f|e>x*g-2<f|x>g.$$

which lead to Eqs.(4.14a) and (4.14b) in view of Eqs.(4.14c) and (4.14d).□

We can then rewrite Eq.(4.12a) as

$$D(a,p)x=\bar{a}*(p*x)+(x*q)*\bar{a}$$
$$=x*(\bar{a}*p)-2<\bar{a}|e>x*p+2<p|e>(\bar{a}*x)+2<\bar{a}|x>p-2<p|x>\bar{a}$$
$$+(q*\bar{a})*x+2<q|e>(x*\bar{a})-2<\bar{a}|e>(q*x)-2<q|x>\bar{a}+2<\bar{a}|x>q$$
$$=x*(\bar{a}*p)+x*p+2<\bar{a}|x>p-2<p|x>\bar{a}$$
$$+(q*\bar{a})*x+q*x-2<q|x>\bar{a}+2<\bar{a}|x>q.$$

Further,

$$\bar{a}*p=(-e-a)*p=-p+q$$

and

$$q*\bar{a}=q*(-e-a)=-q+p,$$

from which we find

$$D(a,p)x=(x*q+p*x)+2<\bar{a}|x>(p+q)-2<(p+q)|x>\bar{a}. \quad (4.15)$$

Next, we relate this derivation to the standard one ([S.66]) given by

$$d(x,y)=l([x,y]^*)-r([x,y]^*)-3[l(x),r(y)] \quad (4.16a)$$

$$= [l(x), l(y)] + [r(x), r(y)] + [l(x), r(y)], \tag{4.16b}$$

where

$$[x, y]^* \equiv x * y - y * x. \tag{4.16c}$$

We can then similarly find

$$d(f, g)x = \{-2(f * g) - (g * f) + 6 < g|e > g\} * x$$

$$+x*\{2(f*g)+(g*f)-6 < f|e > g\}+6 < f|x > g-6 < g|x > f \tag{4.17}$$

for any $f, g, x \in A^*$. If we choose $f = \bar{a}$ and $g = p + q$ in Eq.(4.17), we find

$$d(\bar{a}, p + q) = 3D(a, p). \tag{4.18}$$

Moreover, if we set

$$u = \alpha\bar{a} + \beta(p + q) + \lambda e$$

$$v = \alpha'\bar{a} + \beta'(p + q) + \lambda' e$$

for arbitrary $\alpha, \beta, \lambda, \alpha', \beta', \lambda' \in F$ satisfying

$$\alpha\beta' - \beta\alpha' = 1,$$

then, we can rewrite Eq.(4.18) as

$$d(u, v) = 3D(a, p). \tag{4.18}'$$

In this form, we need not worry about constrained relations between u and v since

$$< u|e >= -\frac{1}{2}\alpha + \lambda, \ < v|e >= -\frac{1}{2}\alpha' + \lambda';$$

thus, we can regard u and v as any generic elements of A^*. Therefore, our derivation $D(a, p)$ represents the standard formula for $Der(A^*)$ if the underlying field F is of characteristic $\neq 2$ and $\neq 3$.

Returning to the discussion of $Aut(A^*)$, the most general formula has been given by Elduque ([E.00], who generalized the earlier result of Jacobson ([J.58]).

Proposition 4.6 ([E.00])

Let A^* be a Cayley-Dickson algebra with the unit element e.
(i) For any $a_1, \cdots, a_r \in A^*$ and for any positive integer r satisfying

$$a_1 * (a_2 * (\cdots (a_{r-1} * a_r)) = e = ((a_r * a_{r-1}) \cdots) * a_1, \tag{4.19}$$

we have

$$Auto(A^*) = \{\prod_{i=1}^{r} l(a_j), \forall r \in Z,\ \forall a_i \in A^*\} \tag{4.20a}$$

$$= \{\prod_{i=1}^{r} r(a_i),\ \forall r \in Z,\ \forall a_i \in A^*\} \tag{4.20b}$$

$$= \{\prod_{i=1}^{r} l(a_i)r(a_i),\ \forall r \in Z,\ \forall a_i \in A^*\}. \tag{4.20c}$$

(ii) Let the underlying field F be of characteristic $\neq 2$ and let $a_1, a_2, \cdots, a_r$ with $r = 2s =$ even satisfy

$$< e|a_i >= 0,\ a_1 * (a_2 * (\cdots (a_{r-1} * a_r))) = e \tag{4.21}$$

instead of Eq.(4.19). Then, $Aut(A^*)$ is still expressed as Eq.(4.20). In particular, $Aut(A^*)$ is an inner automorphosm.

Remark 4.7

The relationship between this formula and the invariant subgroup G of $Auto(A)$, generated by $\sigma(a)$ in Theorem 4.1, is not clear. In the course of its study, we have found the following rather peculiar automorphism $\sigma \in Aut(A^*)$ satisfying $\sigma^2 = 2\sigma - 1$ from Eqs.(4.20) for the case of $r = 3$. More generally, we note

Theorem 4.8

Let A^* be any algebra over a field F of characteristic $\neq 2$, which need not be a Hurwitz algebra. If $\sigma \in Aut(A^*)$ satisfies $\sigma^2 = 2\sigma - 1$, then $d = \sigma - 1$ is a derivation of A^* satisfying $dd = 0$. Conversely, if

$d \in Der(A^*)$ satisfies $dd = 0$, then $\sigma = d + 1$ is an automorphism of A^* satisfying $\sigma^2 = 2\sigma - 1$.

Proof

Suppose that $\sigma \in \mathrm{Auto}(A^*), \sigma \neq 1$ satisfies $\sigma^2 = 2\sigma - 1$. We then calculate

$$(2\sigma - 1)(x * y) = \sigma^2(x * y) = \sigma\{(\sigma x) * (\sigma y)\} = (\sigma^2 x) * (\sigma^2 y)$$

$$= (2\sigma - 1)x * (2\sigma - 1)y = 4(\sigma x) * (\sigma y) - 2(\sigma x) * y - (2x)(\sigma y) + x * y$$

$$= 4\sigma(x * y) - 2(\sigma x) * y - 2x * (\sigma y) + x * y,$$

which is rewritten as

$$d(x * y) = (dx) * y + x * (dy)$$

for $d = \sigma - 1$, which also satisfies $dd = (\sigma - 1)^2 = 0$.

Conversely, if $d \in \mathrm{Der}(A^*)$ satisfies $dd = 0$, then we have

$$0 = dd(x * y) = d\{dx * y + x * dy\} =$$

$$(ddx) * y + (dx) * (dy) + (dx) * (dy) + x * (ddy);$$

thus, we obtain $(dx) * (dy) = 0$. We now calculate

$$\sigma(x * y) = (d + 1)(x * y) = (dx) * y + x * (dy) + x * y$$

$$= (d + 1)x * (d + 1)y = \sigma x * \sigma y.$$

Moreover, $\sigma^2 = (d + 1)^2 = 2d + 1 = 2\sigma - 1$.□

For any integer $n = 0, \pm 1, \pm 2, \cdots$, we then find

$$\sigma^n = n(\sigma - 1) + 1,$$

which leads to $\sigma^n \sigma^m = \sigma^{n+m}$ for any integers n and m. If the field F is of characteristic p, then this yields $\sigma^p = 1$; thus, σ generates a cyclic group Z_p. On the other hand, if F is of characteristic zero, then the generated group is an infinite-dimensional one.

Returning to the discussion of Proposition 4.6, the case of $r = 2$ in Eq.(4.20) gives only the trivial automorphism $\sigma = 1$. To see this, we note that Eq.(4.19) implies

$$a_1 * a_2 = e = a_2 * a_1,$$

and we have $a_2 =< a_2|a_2 > \bar{a}_1$ with $< a_1|a_1 >=< a_2|a_2 >= 1$, giving

$$\sigma = l(a_1)l(a_2) =< a_2|a_2 > l(a_1)l(\bar{a}_1) =< a_2|a_2 >< a_1|a_1 > Id = Id.$$

Similarly, $r(a_1)r(a_2) = 1$.

We consider the case of $r = 3$, which gives $\sigma \in Aut(A^*)$ satisfying $\sigma^2 = 2\sigma - 1$, as we will sketch below. In this third case, Eq.(4.19) becomes

$$a_1 * (a_2 * a_3) = e = (a_3 * a_2) * a_1. \tag{4.22}$$

For simplicity, we normalize the a_j to satisfy

$$< a_1|a_1 >=< a_2|a_2 >=< a_3|a_3 >= 1 \tag{4.23}$$

by extending the field F, if necessary. Then Eq.(4.22) is equivalent to the validity of

$$a_j * a_{j+1} = a_{j+1} * a_j = \overline{a_{j+2}} \tag{4.24}$$

with $a_{j\pm 3} = a_j$. In particular, $(a_1, a_2, a_3) \in \sum$ (see Eq.(2.28)). Moreover, since $a_j * a_{j+1} = a_{j+1} * a_j$, the quadratic relation

$$x * \bar{y} + y * \bar{x} = 2 < x|y > e$$

together with Eq.(4.23) gives

$$a_j + \varepsilon_{j+2}a_{j+1} + \varepsilon_{j+1}a_{j+2} = (2\varepsilon_j + < a_{j+1}|a_{j+2} >)e, \tag{4.25a}$$

where we here set

$$\varepsilon_j =< e|a_j > \tag{4.25b}$$

for $j = 1, 2, 3$. Therefore, if

$$\mathrm{Det}\begin{bmatrix} 1 & \varepsilon_3 & \varepsilon_2 \\ \varepsilon_3 & 1 & \varepsilon_1 \\ \varepsilon_2 & \varepsilon_1 & 1 \end{bmatrix} = 1 - (\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2) + 2\varepsilon_1\varepsilon_2\varepsilon_3 \tag{4.26}$$

is not zero, we must have $a_j = \lambda_j e$ $(j = 1, 2, 3)$ for some $\lambda_j \in F$ satisfying $\lambda_1\lambda_2\lambda_3 = 1$. However, we then calculate

$$\sigma = l(a_1)l(a_2)l(a_3) = r(a_1)r(a_2)r(a_3) = \prod_{r=1}^{3} l(a_r)r(a_r) = 1$$

to be a trivial automorphism. Hence, to obtain a nontrivial σ, we must suppose

$$1 - (\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2) + 2\varepsilon_1\varepsilon_2\varepsilon_3 = 0. \tag{4.27}$$

Furthermore, from Eq.(4.25a), we obtain

$$(1 - \varepsilon_2^2)a_1 = (\varepsilon_1\varepsilon_2 - \varepsilon_3)a_2 + \{2(\varepsilon_1 - \varepsilon_2\varepsilon_3) + < a_2|a_3 > -\varepsilon_2 < a_1|a_2 >\}e.$$

Then, if $\varepsilon_2^2 \neq 1$, using this together with Eq.(4.25a) for $j = 3$, we can express both a_1 and a_3 as linear combinations of a_2 and e, which generate an associative algebra together with any $x \in A^*$ by Artin's theorem ([S.66]). We then again find

$$\sigma = l(a_1)l(a_2)l(a_3) = l((a_1 * a_2) * a_3) = l(e) = 1$$

to be trivial. To obtain a nontrivial σ, we must then have the relation $\varepsilon_1^2 = \varepsilon_2^2 = \varepsilon_3^2 = 1$, $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$ from Eq.(4.26).

Remark 4.9. Let $\varepsilon_j \in F (j = 1, 2, 3)$ be constants satisfying

$$\varepsilon_1^2 = \varepsilon_2^2 = \varepsilon_3^2 = 1, \ \varepsilon_1\varepsilon_2\varepsilon_3 = 1. \tag{4.28}$$

Also, let $b_j \in A^*$ $(j = 1, 2, 3)$ satisfy

(i)

$$< b_i|e >= 0 =< b_i|b_j > \tag{4.29a}$$

(ii)

$$b_i b_j = 0 \tag{4.29b}$$

(iii)

$$\varepsilon_1 b_1 + \varepsilon_2 b_2 + \varepsilon_3 b_3 = 0 \tag{4.29c}$$

for $i, j = 1, 2, 3$. Then, $a_j \in A^*$ defined by

$$a_j = b_j + \varepsilon_j e \tag{4.30}$$

satisfies Eq.(4.24). Moreover, $\sigma \in End(A^*)$ given by

$$\sigma = l(a_1)l(a_2)l(a_3) \tag{4.31}$$

is an automorphism of A^* satisfying $\sigma^2 = 2\sigma - 1$. In this case, we can also rewrite σ as

$$\sigma = 1 + \varepsilon_3 l(b_1)l(b_2) = 1 + \varepsilon_2 l(b_3)l(b_1) = 1 + \varepsilon_1 l(b_2)l(b_3). \tag{4.32}$$

We will not go into the detail of this proof. We also similarly find

$$r(a_1)r(a_2)r(a_3) = \prod_{i=1}^{3} l(a_i)r(a_i) = \sigma.$$

The next question is whether or not nontrivial b_j satisfying Eqs.(4.29) exist. The answer is affirmative if A^* is a split Cayley algebra, although we will not go into detail; however, we note that we may find the elements b_j by utilizing Eqs.(6.11) and (6.12) in ([O.95]).

Finally, we simply note that since $a = (a_1, a_2, a_3) \in \Sigma$ in this case of $r = 3$, we can construct an element of $Trig(A)$ (but not $Trig(A^*)$) by Theorem 2.5.

Next, let us consider the case of $r = 4$, noting that the special choice of $a_4 = e$ reduces the problem to that for the case of $r = 3$. More generally, the case of $r = n$ for an integer n can be regarded as a special instance of $r = n + 1$ with the choice of $a_{n+1} = e$. However, the analysis for a general case is complicated and we did not succeed in finding a relationship between Theorem 4.1 and Proposition 4.6. In future, we would like to consider for the relation between triality and automorphism groups in the general case of nonassociative algebras from this viewpoint.

5 Examples of Triality Groups for Some Nonassociative Algebras with Involution

In this section, we will give some examples of the triality group for nonassociative algebras over a field F of characteristic $\neq 2$ and $\neq 3$ with involution other than the symmetric composition algebra.

Example 5.1 (Matrix Algebra)

Let $V = M(n, F)$ be the set consisting of all $n \times n$ matrices over the field F. For any $x, y \in M(n, F)$, the matrix product, which we denote as $x * y$, is associative and we write

$$x * (y * z) = (x * y) * z := x * y * z. \tag{5.1}$$

Moreover, for the transpose matrix ${}^t x$ of any $x \in M(n, F)$, we set

$$\bar{x} = {}^t x. \tag{5.2}$$

Then $x \to \bar{x}$ is an involution map, thus, the matrix algebra V has the structure of an involutive algebra, denote by A^* it induced from V. That is, A^* is a unital involutive associative algebra(i.e., $\overline{x * y} = \overline{y} * \overline{x}$) with respect to the product $x * y$. In particular, it is structurable ([K-O.15]).

Note that here the $n \times n$ unit matrix e is the unit element of A^*. We introduce a subset of A^* as

$$A_0^* = \{x | \bar{x} * x = x * \bar{x} = e,\ x \in A^*\}. \tag{5.3}$$

For any three $a_j \in A_0^*$ $(j = 1, 2, 3)$, we introduce $\sigma_j(a) \in \mathrm{End}(A^*)$ as

$$\sigma_j(a)x := a_j * x * \bar{a}_{j+1}\ (j = 1, 2, 3), \tag{5.4}$$

where the indices over j are defined modulo 3, i.e., $a_{j\pm 3} = a_j$. It is easy to see the validity of

$$\sigma_j(a)\sigma_j(\bar{a})x = a_j * \overline{a}_j * x * a_{j+1} * \overline{a}_{j+1} * x = x \tag{5.5a}$$

$$\overline{\sigma_j(a)}x = a_{j+1} * x * \bar{a}_j \tag{5.5b}$$

$$\overline{\sigma_j(a)}(x * y) = (\sigma_{j+1}(a)x) * (\sigma_{j+2}(a)y). \tag{5.5c}$$

Introducing the conjugate algebra $(A^*)^*$ $(= A)$ of A^*, with the bilinear product xy as

$$xy = \overline{x * y} = \bar{y} * \bar{x}, \tag{5.6}$$

Eqs.(5.5) are rewritten as

$$\sigma_j(a)x = a_{j+1}(a_j x) = (x\bar{a}_{j+1})\bar{a}_j \tag{5.7a}$$

$$\sigma_j(a)(xy) = \{(\sigma_{j+1}(a)x)\}\{(\sigma_{j+2}(a)y)\}. \tag{5.7b}$$

We note that A is no longer associative but para-associative with the para-associative law of

$$\bar{z}(xy) = (yz)\bar{x}. \tag{5.8}$$

From Eqs.(5.5a) and (5.7b), we find

$$\sigma(a) = (\sigma_1(a), \sigma_2(a), \sigma_3(a)) \in \mathrm{Trig}(A). \tag{5.9}$$

Furthermore, Eq.(5.7a) gives

$$\sigma_j(a) = L(a_{j+1})L(a_j) = R(\bar{a}_j)R(\bar{a}_{j+1}), \tag{5.10}$$

which has the same structures as Eq.(2.29) in Theorem 2.5 for the symmetric composition algebra.

For the corresponding local triality case, let

$$p = (p_1, p_2, p_3) \in (A^*)^3, \ \bar{p}_j = -p_j \ (j = 1, 2, 3) \tag{5.11}$$

and define $d_j(p) \in \mathrm{End}\ A^*$ by

$$d_j(p)x = p_j * x - x * p_{j+1}. \tag{5.12}$$

We then have

$$\overline{d_j(p)}x = x * \bar{p}_j - \bar{p}_{j+1} * x = p_{j+1} * x - x * p_j \tag{5.13a}$$

$$\overline{d_j(p)}(x * y) = (d_{j+1}(p)x) * y + x * (d_{j+2}(p)y), \tag{5.13b}$$

and hence

$$d_j(p)(xy) = (d_{j+1}(p)x)y + x(d_{j+2}(p)y). \tag{5.14}$$

This implies that $d(p) = (d_1(p), d_2(p), d_3(p)) \in s \circ Lrt(A)$.

Thus, this example of matrix algebra may be regarded as a generalization of the correspondence as follows:

$$a * x * a^{-1} \ (\textit{automorphism}) \Longleftrightarrow p * x - x * p \ (\textit{derivation}),$$

where a is orthogonal and p is alternative, that is, by means of the well-known Cayley transformation of a linear Lie group, this correspondence

means that $a = (Id - p) * (Id + p)^{-1}$ (assuming that it is well-defined). However, we will not go into detail.

It seems that this local triality relation represented by Eq.(5.14) may be regarded as a version for matrix algebras that corresponds with the well-known "principle of triality" for a Cayley algebra ([S.66]). Similarly, the relation Eq.(3.6) of Theorem 3.2 may be regarded as a local triality relation (the principle of triality) for a symmetric composition algebra.

Example 5.2 (Para-Zorn Matrix Algebra)

Let B be an algebra over a field F with a bilinear form $< \cdot | \cdot >$ and let

$$A = \begin{pmatrix} F & B \\ B & F \end{pmatrix} \tag{5.15}$$

be the Zorn vector matrix with its generic element $X \in A$ given by

$$X = \begin{pmatrix} \alpha & x \\ y & \beta \end{pmatrix} \tag{5.16}$$

for $\alpha, \beta \in F$ and $x, y \in B$. We introduce a bilinear product in A as

$$X_1 X_2 = \begin{pmatrix} \alpha_1 & x_1 \\ y_1 & \beta_1 \end{pmatrix} \begin{pmatrix} \alpha_2 & x_2 \\ y_2 & \beta_2 \end{pmatrix} =$$

$$\begin{pmatrix} \beta_1\beta_2 + < y_1 | x_2 >, & \alpha_1 x_2 + \beta_2 x_1 + k y_1 y_2 \\ \alpha_2 y_1 + \beta_1 y_2 + k x_1 x_2, & \alpha_1\alpha_2 + < x_1 | y_2 > \end{pmatrix} \tag{5.17}$$

for $k \in F$.

If B is involutive with the involution map $x \to \bar{x}$ satisfying

$$\overline{xy} = \bar{y}\bar{x}, \ < \bar{x} | \bar{y} > = < y | x >, \tag{5.18}$$

then A also becomes involutive since

$$X \to \bar{X} = \begin{pmatrix} \beta & \bar{x} \\ \bar{y} & \alpha \end{pmatrix}, \tag{5.19}$$

and its conjugate algebra A^* with the bilinear product

$$X_1 * X_2 = \overline{X_1 X_2} = \bar{X}_2 \bar{X}_1 \tag{5.20}$$

represents the standard Zorn vector matrix algebra ([K-O.14]).

Also we note that if the involution of B is the identity map, then B is commutative. Moreover, we assume that if B is the exceptional Jordan algebra with $dim\ B = 27$, then for $k = 2$, A^* is a structurable algebra ([A.78],[K-O.15]).

If the involution of B is anticommutative (i.e., $\bar{x} = -x$) and B satisfies the conditions $x(yz) =< x|y > z - < x|z > y$ and $< x|yz >=< y|zx >=< z|xy >$, then it is known that for $k = 1$, A^* is an alternative algebra ([K-O.15]).

On the other hand, by utilizing the Zorn vector matrices, we have considered the construction of simple Lie algebras or superalgebras from the concept of triple systems with a ternary product $(X_1X_2X_3)$ in A^*; however, we will not go into details (for example, see [K-O.15] and references therein).

Next, for any nonzero $\lambda \in F$, we introduce $\rho_j(\lambda) \in End\ (A)$ $(j = 1, 2, 3)$ as

$$\rho_1(\lambda)X = \begin{pmatrix} \lambda\alpha, & \lambda x \\ \lambda^{-1}y, & \lambda^{-1}\beta \end{pmatrix} \tag{5.21a}$$

$$\rho_2(\lambda)X = \begin{pmatrix} \lambda\alpha, & \lambda^{-1}x \\ \lambda y, & \lambda^{-1}\beta \end{pmatrix} \tag{5.21b}$$

$$\rho_3(\lambda)X = \begin{pmatrix} \lambda^{-2}\alpha, & x \\ y, & \lambda^2\beta \end{pmatrix}. \tag{5.21c}$$

We can then show the following.

Theorem 5.3

Under the assumption in Ex.5.2, we have

(i)

$$\rho_j(\lambda)(XY) = (\rho_{j+1}(\lambda)X)(\rho_{j+2}(\lambda)Y) \tag{5.22a}$$

for any $X, Y \in A$ *and for any* $j = 1, 2, 3$ *with* $\rho_{j\pm3}(\lambda) = \rho_j(\lambda)$.

(ii)

$$\rho_j(\mu)\rho_j(\nu) = \rho_j(\mu\nu),\ \rho_j(1) = 1 \tag{5.22b}$$

(iii)

$$\rho_j(\mu)\rho_k(\nu) = \rho_k(\nu)\rho_j(\mu) \tag{5.22c}$$

(iv)

$$\rho_1(\lambda)\rho_2(\lambda)\rho_3(\lambda) = 1 \tag{5.22d}$$

(v) If B is involutive, then

$$\overline{\rho_j(\lambda)} = \rho_{3-j}(\lambda^{-1}) \tag{5.22e}$$

for any $j, k = 1, 2, 3$ and for any nonzero $\mu, \nu \in F$.

Proof

Since the calculations are straightforward, we will verify only a few of these statements. First,

$$(\rho_1(\lambda)X_1)(\rho_2(\lambda)X_2) = \begin{pmatrix} \lambda\alpha_1, & \lambda x_1 \\ \lambda^{-1}y_1, & \lambda^{-1}\beta_1 \end{pmatrix} \begin{pmatrix} \lambda\alpha_1, & \lambda^{-1}x_2 \\ \lambda y_2, & \lambda^{-1}\beta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^{-2}\beta_1\beta_2 + \lambda^{-2}(y_1|x_2), & \alpha_1 x_2 + \beta_2 x_1 + k y_1 y_2 \\ \alpha_2 y_1 + \beta_1 y_2 + k x_1 x_2, & \lambda^2\alpha_1\alpha_2 + \lambda^2(x_1|y_2) \end{pmatrix} = \rho_3(\lambda)(X_1X_2),$$

which proves the case of $j = 3$ for Eq.(5.22a). Similarly, we calculate

$$(\rho_2(\lambda)X_1)(\rho_3(\lambda)X_2) = \begin{pmatrix} \lambda\alpha_1, & \lambda^{-1}x_1 \\ \lambda y_1, & \lambda^{-1}\beta_1 \end{pmatrix} \begin{pmatrix} \lambda^{-2}\alpha_2, & x_2 \\ y_2, & \lambda^2\beta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda\beta_1\beta_2 + \lambda(y_1|x_2), & \lambda\alpha_1 x_2 + \lambda\beta_2 x_1 + \lambda k y_1 y_2 \\ \lambda^{-1}\alpha_2 y_1 + \lambda^{-1}\beta_1 y_2 + \lambda^{-1}k x_1 x_2, & \lambda^{-1}\alpha_1\alpha_2 + \lambda^{-1}(x_1|y_2) \end{pmatrix}$$

$$= \rho_1(\lambda)(X_1X_2)$$

for the case of $j = 1$.

We also note, for example,

$$\rho_2(\lambda)\rho_3(\lambda)X = \rho_2(\lambda)\begin{pmatrix} \lambda^{-2}\alpha, & x \\ y, & \lambda^2\beta \end{pmatrix} = \begin{pmatrix} \lambda(\lambda^{-2}\alpha), & \lambda^{-1}x \\ \lambda y, & \lambda^{-1}(\lambda^2\beta) \end{pmatrix}$$

$$= \rho_1(\lambda^{-1})X,$$

implying that

$$\rho_1(\lambda)\rho_2(\lambda)\rho_3(\lambda)X = \rho_1(\lambda)\rho_1(\lambda^{-1})X = X.$$

Finally, if B is involutive, then we compute

$$\overline{\rho_1(\lambda)}\bar{X} = \overline{\rho_1(\lambda)X} = \begin{pmatrix} \lambda^{-1}\beta, & \lambda\bar{x} \\ \lambda^{-1}\bar{y}, & \lambda\alpha \end{pmatrix}.$$

On the other hand,

$$\rho_2(\lambda^{-1})\bar{X} = \rho_2(\lambda^{-1}) \begin{pmatrix} \beta & \bar{x} \\ \bar{y} & \alpha \end{pmatrix} = \begin{pmatrix} \lambda^{-1}\beta, & \lambda\bar{x} \\ \lambda^{-1}\bar{y}, & \lambda\alpha \end{pmatrix},$$

and hence $\overline{\rho_1(\lambda)} = \rho_2(\lambda^{-1})$. We can similarly verify other relations.□

Remark 5.4

Let us set

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ g(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix},\ h(\lambda) = \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \lambda \end{pmatrix}. \tag{5.23}$$

Then, we have

$$eX = Xe = \begin{pmatrix} \beta, & x \\ y, & \alpha \end{pmatrix} \tag{5.24}$$

for $X = \begin{pmatrix} \alpha & x \\ y & \beta \end{pmatrix}$, and $eg(\lambda) = h(\lambda), g(\lambda)h(\lambda) = e, h(\lambda)e = g(\lambda)$, as in the property of $a_{j+2} = a_j a_{j+1}$ in section 2, provided $< e|e >:= det\ e = 1, < g(\lambda)|g(\lambda) >:= det\ g(\lambda) = 1, < h(\lambda)|h(\lambda) >:= det\ h(\lambda) = 1$.

Moreover, we can express $\rho_j(\lambda)$ as

$$\rho_1(\lambda) = L(e)L(g(\lambda)) = R(e)R(h(\lambda)) \tag{5.25a}$$

$$\rho_2(\lambda) = L(h(\lambda))L(e) = R(g(\lambda))R(e) \tag{5.25b}$$

$$\rho_3(\lambda) = L(g(\lambda))L(h(\lambda)) = R(h(\lambda))R(g(\lambda)), \tag{5.25c}$$

which have structures similar to Eqs.(2.29) for the symmetric composition algebra.

This implies that $\rho(\lambda) = (\rho_1(\lambda), \rho_2(\lambda), \rho_3(\lambda)) \in Trig\ (A)$.

Next, let $\pi \in End\ (A)$ be given by

$$\pi : \begin{pmatrix} \alpha & x \\ y & \beta \end{pmatrix} \to \begin{pmatrix} \alpha & y \\ x & \beta \end{pmatrix}, \tag{5.26}$$

which satisfies

(i)

$$\pi(XY) = (\pi X)(\pi Y) \tag{5.27a}$$

(ii)

$$\pi^2 = 1 \tag{5.27b}$$

(iii)

$$\pi\rho_j(\lambda)\pi^{-1} = \rho_{3-j}(\lambda) = \bar{\rho}_j(\lambda^{-1}). \tag{5.27c}$$

Hence, the group generated by $\rho(\lambda) = (\rho_1(\lambda), \rho_2(\lambda), \rho_3(\lambda))$ and $\pi_0 = (\pi, \pi, \pi)$ will be a subgroup of $Trig\ (A)$.

Moreover, suppose that $\xi,\ \eta \in Epi(A)$ satisfy

(iv)

$$\xi(xy) = (\eta x)(\eta y) \tag{5.28a}$$

(v)

$$\eta(xy) = (\xi x)(\xi y) \tag{5.28b}$$

for any $x, y \in B$. We call the pair (ξ, η) a double automorphism of B. We then find

Proposition 5.5

Under the assumption in Theorem 5.3, if a double automorphism (ξ, η) of B satisfies

$$(\xi x|\eta y) = (x|y), \tag{5.29}$$

then $P \in End(A)$ given by

$$P : X \to \begin{pmatrix} \alpha, & \xi x \\ \eta y, & \beta \end{pmatrix} \tag{5.30a}$$

is an automorphism of the para-Zorn algebra A, i.e., we have

$$P(XY) = (PX)(PY). \tag{5.30b}$$

Proof

We will not go into its proof since it is straightforward.□

For a simple example of a double automorphism, we have $(\xi, \eta) = (\omega, \omega^2)$, where $\omega^3 = 1$.

Finally, let us consider an example of the corresponding local triality Lie-related triple in the para-Zorn matrix algebra A. We introduce $s_j \in End\ A$ for $j = 1, 2, 3$ as

$$s_1 X = \begin{pmatrix} \alpha & x \\ -y, & -\beta \end{pmatrix} \tag{5.31a}$$

$$s_2 X = \begin{pmatrix} \alpha & -x \\ y & -\beta \end{pmatrix} \tag{5.31b}$$

$$s_3 X = \begin{pmatrix} -2\alpha & 0 \\ 0 & 2\beta \end{pmatrix}. \tag{5.31c}$$

Then, by straightforward calculations, we obtain the following.

Proposition 5.6

Under the above assumption, we have
(i)

$$s = (s_1, s_2, s_3) \in s \circ Lrt(A), i.e.,$$

$$s_j(XY) = (s_{j+1}X)Y + X(s_{j+2}Y) \tag{5.32a}$$

(ii)

$$s_1 + s_2 + s_3 = 0 \tag{5.32b}$$

(iii)

$$[s_j, s_k] = 0. \tag{5.32c}$$

(iv) If B is involutive, then

$$\bar{s}_j = -s_{3-j} \tag{5.32d}$$

for any $j, k = 1, 2, 3$, where the indices over j are again defined modulo 3.

Proof

Eqs.(5.32) are obtained by straightforward calculations. For example, from $\bar{s}_j(X) = \overline{s_j(\bar{X})} = -s_{3-j}(X)$, it is easy to show that $\bar{s}_j = -s_{3-j}$ $(j = 0, 1, 2)$.□

To end this section, we see that the three examples considered in this paper with respect to the symmetric composition algebra, matrix algebra, and para-Zorn algebra allow the construction of the "triality group" of the form given in Eq.(2.29). Hence, these examples suggest that there may exist other classes of algebras allowing the construction of triality groups by this method.

On the other hand, for triple systems equipped with a ternary product (xyz) (for example, see [O.95],[K-O.00],[K-M-O.10]), it seems that local and global triality relations may be derived on the basis of the same concept, but this will be considered in future.

Appendix (Simple examples)

In this appendix, we give two simple examples.

Example A Let $\mathbf{C}$ be a complex number with the usual product $x * y$. If we define for $a = (\alpha, \beta, \gamma) \in \mathbf{C}^3$ satisfying $|\alpha| = |\beta| = |\gamma| = 1$,

$$\sigma_1(a)x = \alpha * x * \beta^{-1}$$

$$\sigma_2(a)x = \beta * x * \gamma^{-1}$$

$$\sigma_3(a)x = \gamma * x * \alpha^{-1},$$

where $\sigma_{j\pm 3} = \sigma_j (j = 0, 1, 2)$, then we have

$$\sigma_j(a)(xy) = (\sigma_{j+1}(a)x)(\sigma_{j+2}(a)y)$$

with respect to the new product xy defined by $xy = \overline{x * y}$,
where $\bar{x}$ denotes the conjugation of x.

If we introduce for $a = (\alpha, \beta, \gamma) \in (Im\ \mathbf{C})^3$,

$$d_1(a)x = \alpha * x - x * \beta, d_2(a)x = \beta * x - x * \gamma, d_3(a)x = \gamma * x - x * \alpha,$$

then we obtain

$$d_j(a)(xy) = (d_{j+1}(a)x)y + x(d_{j+2}(a)y).$$

Note that $Aut(\mathbf{C}, xy) \cong S_3$ (as a group) and $Der(\mathbf{C}, xy) \cong \{d \in End\ \mathbf{C} \mid d = \frac{2n}{3}\pi\sqrt{-1},\ \forall n \in \mathbf{Z}(integer)\}$ (as an algebra) with respect the product xy.

Example B Let $\mathbf{H}$ be the quaternion algebra satisfying $i^2 = j^2 = k^2 = -1, i * j = -j * i = k$ with basis $\{1, i, j, k\}$. If we define for $a = (i, j, k) \in (Im\ \mathbf{H})^3$,

$$\sigma_1(a)x = i * x * j^{-1}$$

$$\sigma_2(a)x = j * x * k^{-1}$$

$$\sigma_3(a)x = k * x * i^{-1},$$

where the product $x * y$ is the usual product of $\mathbf{H}$ and $\sigma_{j\pm3} = \sigma_j$, then we have

$$\sigma_j(a)(xy) = (\sigma_{j+1}(a)x)(\sigma_{j+2}(a)y)$$

w.r.t. the new product $xy = \overline{x * y}$, where $\bar{x}$ denotes the involutive conjugation of $x \in \mathbf{H}$.

If we introduce $d_1(a)x = i*x - x*j$, $d_2(a)x = j*x - x*k$, $d_3(a)x = k*x - x*i$, then we obtain

$$d_l(a)(xy) = (d_{l+1}(a)x)y + x(d_{l+2}(a)y), \quad l = l \pm 3$$

w.r.t. the new product xy defined by $xy = \overline{x * y}$.

Concluding Remarks

The above $\mathbf{H}$ has the structure of a symmetric composition algebra with respect to the new product defined by $xy = \overline{x * y}$, Thus, special examples of Σ in Eq.(2.28) are $(i, j, -k)$, $(i, -j, k)$, $(-i, j, k)$. Indeed, from the fact that $\mathbf{H}$ is a symmetric composition algebra with respect to the product xy and $ij = \overline{i * j} = -k$, etc., we obtain a simple special case of Theorem 2.5. Furthermore, taking $a = \frac{1}{2}(-1 + \sqrt{3}\ i)$, then we have $a * \bar{a} = 1$, $aa = a$, $< a|a >= 1$, because $aa = \frac{1}{4}\overline{(-1 + \sqrt{3}\ i) * (-1 + \sqrt{3}\ i)} = a$ and so a is an idempotent element, hence it satisfies the assumption of Theorem 4.1. That is, if we set $\sigma(a) = R(a)R(a)$, then $\sigma(a)$ is an automorphism, i.e.,

$$\sigma(a)(xy) = (\sigma(a)x)(\sigma(a)y).$$

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FOUR INDEPENDENT EIGENVECTORS AND THEIR PROPERTIES FOR THE PROBLEM OF ACOUSTIC 4P-SH-WAVE PROPAGATION INCORPORATING GRAVITATIONAL PHENOMENA

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Abstract

This theoretical investigation has obtained four independent pairs of main eigenvectors called the *E*-, *H*-, *G*-, and *F*-eigenvectors. Their properties are also demonstrated. The obtained eigenvectors together with the eigenvalues represent the solutions of the coupled equations of motion when the shear-horizontal (SH) acoustic wave propagation in the 6 *mm* solids is coupled with the electrical, magnetic, gravitational, and cogravitational potentials, i.e. 4P-SH-waves. To obtain suitable forms of the eigenvectors is very important because they form the final solutions of the propagation velocities, with which the acoustic waves propagate in the solids. It is expected that their proper forms can demonstrate dominating factors causing the acoustic wave propagation, i.e. the physical properties of the wave processes. To know the acoustic wave propagation velocities in the solids is one of very important engineering problems in the signal processing technology. This allows the constitution of various technical devices such as filters, sensors, delay lines, energy harvesting and wireless devices, etc. The successful development of the theoretical, mathematical, experimental, engineering, and technological directions of investigations of the four-potential waves and their applications can result in the instant interplanetary (interstellar and even intergalactic) communications, for instance, the instant interplanetary Internet. This is possible because the evaluated speeds of the recently discovered fast four-potential waves propagating in a vacuum can be thirteen orders faster than the speed of light, i.e. the speed of the electromagnetic waves representing two-potential waves.

Keywords: Transversely isotropic continuous media; Acoustic wave propagation; Gravitational effects; Four potential coupling problem.

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1 Introduction

In the jubilee year 2016, it was developed the theory [1] that incorporates the mechanical, electrical, magnetic, gravitational, and cogravitational subsystems in the wave motion process. As a result, the new four-potential shear-horizontal surface acoustic wave (4P-SH-SAW) [1] was discovered when the acoustic wave propagates in the transversely isotropic material of symmetry class 6 *mm*. It was a centenary jubilee of the prediction of the existence of the gravitational waves by Albert Einstein in 1916 [2]. To this jubilee, a very large international research team has experimentally proved the existence of the gravitational waves. Their experimental results [3] published in 2016 have soundly confirmed the existence of the theoretically predicted gravitational waves.

Developed theory [1] uses this experimental fact of the propagation of the gravitational waves in the free space (vacuum) at the speed of light. It is well-known that the electromagnetic waves also propagate in a vacuum at the speed of light. The speed of light was precisely measured only in the 1970s and now can be found in any modern reference book on physics. For instance, the measured value of the speed of light can be found in popular reference book [4]. So, it is possible to write down the following value of the speed of light: $C_L = (\varepsilon_0\mu_0)^{-1/2}\sim(\gamma_0\eta_0)^{-1/2} = 2.99792458 \times 10^8$ [m/s], where ε_0, μ_0, γ_0, and η_0 are the electric constant, magnetic constant, gravitic constant, and cogravitic constant for a vacuum, respectively. A gravitational and electromagnetic analogy was first stated by Heaviside [5] and then developed by many researchers. The famous book by Jefimenko [6] operates with gravitation and cogravitation to develop Newton's theory of gravitation to its physical and mathematical conclusion. So, the purely gravitational analogy to the theory of electromagnetism is now called the theory of the gravitoelectromagnetism. It is expected that the gravitic constant γ_0 and the cogravitic constant η_0 for a vacuum can be precisely measured in this century and then recorded in physics reference books.

Created theory [1] relating to numerical experiment takes into account the fact that the speed of the electromagnetic waves in both a solid and a vacuum is five orders faster than the speed of any acoustic wave in a solid. As a result, a quasi-static approximation is used in the theory of acoustic wave propagation in continuous solid media. In the solids, the slow speed of the new 4P-SH-SAW [1] can naturally depend on the speed of the electromagnetic wave. However, it also depends on the speed of new fast waves that can be thirteen orders faster

than the speed of light. These new fast waves can also propagate in a vacuum with the following speeds: $\Lambda_{01} = (\zeta_0\lambda_0)^{-1/2} \to 10^{13}C_L$ and $\Lambda_{02} = (\xi_0\beta_0)^{-1/2} \to 10^{13}C_L$, where the vacuum parameters ζ_0, λ_0, ξ_0, and β_0 are called the gravitoelectric, cogravitomagnetic, cogravitoelectric, and gravitomagnetic constants, respectively. These vacuum parameters relating to the gravitational phenomena must be also measured and recorded in any modern reference book on physics.

The existence of the new fast waves propagating in a vacuum with the speeds Λ_{01} and Λ_{02} can be true because the famous Russian astronomer and astrophysicist Kozyrev [7] has created his own tools, with which he has observed true positions of stars from his observatory. It is obvious that astronomers use optic tools to observe stars. As a result, they can for instance observe a star distant from the Earth on several light years at its position it was several light years ago. There are a lot of stars distant from the Earth on 10, 100, 1000 light years. Therefore, any optic tools for observation of stars cannot record true positions of the stars. To observe the true position of any star, Kozyrev [7] has first used some optic tools to observe a star. After that, he has calculated the true position of the star and used his own tools to observe this star at the true position. This means that there is something that can propagate in a vacuum significantly faster than the speed of light. To explain this phenomenon, Kozyrev has developed his own theory called the theory of time.

Kozyrev is not the single researcher who has found that there is something significantly faster than the speed of light. The French mathematician Pierre-Simon Laplace [8], also known as the French Newton, has evaluated the stability of the Solar system. He has investigated the motion of space bodies in the Solar system and concluded that gravity must propagate millions of times faster than light, since otherwise violation of Newton's law of universal gravitation would be observed. It is worth noting that Newton's law of universal gravitation assumes an infinitely large speed of gravity, i.e. the gravity can instantly propagate. However, the first attempt to explain Newton's law of gravitation was done by Nicolas Fatio de Duillier in 1690 and by Georges-Louis Le Sage in 1748. This is a kinetic theory of gravity originally proposed by Fatio and later developed by Le Sage. The theory proposed a mechanical explanation for Newton's gravitational force in terms of streams of tiny unseen particles impacting all material objects from all directions [9]. Concerning the movements of planets in the Solar system in the fluid consisting of the particles (i.e. in the

free space), Le Sage [9] has used an argument to demonstrate that the speed of the fluid must be thirteen orders faster than the speed of light. To review all existing evaluations of the speed of gravity and possible gravitational phenomena does not represent the main purpose of this study. Therefore, it is necessary to return to modern theory [1] because the thermodynamic functions and the thermodynamic variables were used to develop this theory.

The hottest application of the new fast waves propagating in a vacuum at the speeds A_{01} and A_{02} is the instant interplanetary communication [10, 11] inside the Solar system, for instance, the space Internet. The possible applications proposed in paper [10] for the new fast waves propagating in the continuous medium such as a vacuum is the natural consequence from theory [1] developed for the continuous medium such as a solid. Theory [1] considers the acoustic wave propagation in the smart solid materials such as the piezoelectromagnetics (magnetoelecroelastics) with taking into account the gravitational phenomena. In this case, the gravitational phenomena significantly complicate the theoretical developments compared with the case of pure piezoelectromagnetics without the phenomena [12, 13, 14]. However, it is necessary to treat the gravitational phenomena because the propagation of acoustic waves means energy transmission between the wave generator and the wave detector. One century ago Einstein has postulated that any kind of energy (and any change in energy) is coupled with gravitation. Therefore, theory [1] treats the gravitational and cogravitational subsystems in addition to the mechanical, electrical, and magnetic subsystems. As a result, the wave motions are caused by exchange interactions (i.e. some energetic exchanges) among the subsystems. It is expected that such energetic system consisting of five subsystems can be more reach than the purely piezoelectromagnetic system consisting of only three subsystems such as the mechanical, electrical, and magnetic subsystems. It is natural because piezoelectromagnetics with (without) taking into account the gravitational phenomena can be used instead of pure piezoelectrics.

Today, piezoelectrics are utilized in various technical devices such as filters, (chemical and biological) sensors, delay lines, energy harvesting and wireless devices, etc. They are also used in the cutting-edge technologies of mobile communication and wireless sensors [15, 16]. The piezoelectric materials are also studied for applications in energy harvesting devices [16, 17, 18, 19]. However, smart piezoelectromagnetic materials are more preferable for constitution of the wireless devices [20, 21, 22] in comparison with the

conventional piezoelectrics. It is also expected that the piezogravitocogravitoelectromagnetic materials [1] with the mechanical, electrical, magnetic, gravitational, and cogravitational subsystems can be more reach regarding the energy harvesting and energy conversion. Recently, Füzfa [23] has found a method to study interactions of the electrical or magnetic subsystem with the gravitational or cogravitational subsystem at the laboratory conditions on the Earth instead of very expensive space experiments. The theoretical, mathematical, experimental, engineering, and many relative investigations can be useful for the development of infrastructure including the technical devices for generating, receiving, and signal processing of the new fast waves for the new era of the instant interplanetary communication.

The theoretical investigations were initiated in pioneer paper [1]. However, this created theory has released some mathematical problems that must be also resolved. One of the most important mathematical problems is the finding of proper eigenvectors because their forms can result in final solutions for the propagating velocities of the acoustic waves. Theoretical work [24] has demonstrated only half of the twenty-four possible eigenvectors that can be obtained in the framework of theory [1]. However, all of them can be reduced to four main independent eigenvectors. To obtain these four eigenvectors called the *E*-, *H*-, *G*-, and *F*-eigenvectors and their properties is the main purpose of this study. The theoretical part and mathematical methods are very important even in the case of the acoustic wave propagation in the pure piezoelectrics. To find propagation velocities in different structures represents an engineering problem. There is famous book by Dieulesaint and Royer [25] concerning applications of the elastic waves in the piezoelectric solids to signal processing. Therefore, original theory [1] represents a rudiment for further development of the theory of the acoustic wave propagation in the piezogravitocogravitoelectromagnetic solids guided by the surface [1], interface [26], and plate (thin film) [27]. In all these cases, it is very important to know eigenvalues and eigenvectors representing the solutions of the corresponding coupled equations of motion.

2 The equations of motion and solutions

To construct the coupled equations of motion, it is necessary first of all to treat the suitable thermodynamic functions and thermodynamic variables for an adiabatic process in solid continuous media. For the mechanical, electrical, magnetic, gravitational, and cogravitational subsystems, the thermodynamic

functions are the mechanical stress, electrical induction, magnetic induction, gravitational induction, and cogravitational induction. The thermodynamic variables are the mechanical strain, electrical field, magnetic field, gravitational field, and cogravitational field. So, this is a multidisciplinary research arena. The mechanical parameters came from the theory of elasticity, the electrical and magnetic parameters from the theory of electromagnetism, and the gravitational and cogravitational parameters from the theory of gravitoelectromagnetism. There is some analogy between the theories of gravitation and the electromagnetism [5]. As a result, some analogy can be found between the theories of electricity and gravitation (i.e. gravitoelectricity) and between the theories of magnetism and cogravitation [6] (i.e. gravitomagnetism [28]). This part of theory is quite large and the reader can find it in paper [1]. However, it is also necessary to mention that the quasi-static approximation must be applied to obtain the coupled equations of motion in the form of a set of second-order partial differential equations. This approximation is used because the speed of the electromagnetic waves in a solid is close to the speed of light in a vacuum and five orders faster than the speed of any acoustic wave.

To resolve the second-order partial differential equations, it is natural to exploit the plane wave solutions. These solutions naturally transform the differential form of the coupled equations of motion into the tensor form of the equations. In the common case of the acoustic wave propagation in an anisotropic solid, the tensor form can be written down in the following compact expression called the modified Green-Christoffel equation [1]: $(GL_{IJ} - \delta_{IJ}\rho V_{ph})U_I^0 = 0$, where the indices I and J run from 1 to 7 and ρ is the mass density listed in table 1. The phase velocity V_{ph} will be defined below by expression (7). GL_{IJ} stands for the components of the modified symmetric tensor [1] and δ_{IJ} represents the Kronecker delta-function with the following conditions: $\delta_{IJ} = 1$ for $I = J < 4$, $\delta_{IJ} = 0$ for $I \neq J$, and $\delta_{44} = \delta_{55} = \delta_{66} = \delta_{77} = 0$. Also, parameters U_I^0 represent the components of the eigenvector $(U_1^0, U_2^0, U_3^0, U_4^0, U_5^0, U_6^0, U_7^0)$. This tensor form of the coupled equations of motion represents the common problem for determination of the eigenvalues and eigenvectors. This common problem is very complicated and can be numerically resolved. However there are the high-symmetry propagation directions in solid materials with certain crystal symmetries [25, 29] that allow significant simplifications leading to some analytical solutions.

Table 1: The material parameters of the transversely isotropic material (crystal) of symmetry class 6 *mm* and their physical dimensions.

Material parameter	**Symbol [dimension]**
Mass density	ρ [kg/m^3]
Elastic stiffness constant	$C = C_{44} = C_{66}$ [kg/(m×s^2)], [N/m^2]
Piezoelectric constant	$e = e_{16} = e_{34}$ [kg$^{1/2}$/m$^{3/2}$], [C/m^2]
Piezomagnetic coefficient	$h = h_{16} = h_{34}$ [kg$^{1/2}$/(m$^{1/2}$×s)], [N/(A×m)], [T]
Piezogravitic constant	$g = g_{16} = g_{34}$ [kg/m^2]
Piezocogravitic coefficient	$f = f_{16} = f_{34}$ [s^{-1}]
Electric constant	$\varepsilon = \varepsilon_{11} = \varepsilon_{33}$ [s^2/m^2], [F/m]
Magnetic constant	$\mu = \mu_{11} = \mu_{33}$ [-], [N×s^2/C^2], [H/m], [N/A^2], [T×m/A], [Wb/(A×m)], [V×s/(A×m)]
Electromagnetic constant	$\alpha = \alpha_{11} = \alpha_{33}$ [s/m], [N×s/(V×C)]
Gravitic constant	$\gamma = \gamma_{11} = \gamma_{33}$ [kg×s^2/m^3], [kg^2/(N×m^2)]
Cogravitic constant	$\eta = \eta_{11} = \eta_{33}$ [m/kg]
Gravitocogravitic constant	$\vartheta = \vartheta_{11} = \vartheta_{33}$ [s/m]
Gravitoelectric constant	$\zeta = \zeta_{11} = \zeta_{33}$ [kg$^{1/2}$×s^2/m$^{5/2}$], [C×kg/(J×m)]
Cogravitoelectric constant	$\xi = \xi_{11} = \xi_{33}$ [s/(kg$^{1/2}$×m$^{1/2}$)], [m/Wb]
Gravitomagnetic constant	$\beta = \beta_{11} = \beta_{33}$ [kg$^{1/2}$×s/m$^{3/2}$], [T×kg×m/J]
Cogravitomagnetic constant	$\lambda = \lambda_{11} = \lambda_{33}$ [m$^{1/2}$/kg$^{1/2}$], [T×m^3/(C×Wb)]

For a solid of symmetry class 6 *mm*, the high-symmetry propagation directions on the certain cuts of the solid are well-known. According to review paper [30], the sixfold symmetry axis must be directed along the x_2-axis, perpendicular to both the wave propagation direction towards the x_1-axis and the surface normal along the x_3-axis. The rectangular coordinate system $\{x_1, x_2, x_3\}$ and the corresponding wavevector components $\{k_1, k_2, k_3\}$ are used. For this case, the modified Green-Christoffel equation representing a set of seven homogeneous equations splits into two independent sets. The first set of two homogeneous equations with the eigenvector (U_1^0, U_3^0) corresponds to the purely mechanical wave with the in-plane polarization. The second set of five homogeneous equations with the eigenvector $(U_2^0, U_4^0, U_5^0, U_6^0, U_7^0)$ represents the

main interest in this study because here the acoustic wave propagation is coupled with the following four potentials: the electrical φ, magnetic ψ, gravitational Φ, and cogravitational Ψ potentials. This is the case of propagation of the anti-plane polarized four-potential shear-horizontal (4P-SH) acoustic waves.

For this case, it is possible to deal only with the suitable *GL*-tensor components of the modified Green-Christoffel equation [1]. Therefore, the coupled equations of motion describing the 4P-SH-wave propagation coupled with the four potentials can be then expressed by the following five homogeneous equations:

$$\begin{pmatrix} GL_{22}-\rho V_{ph}^2 & GL_{24} & GL_{25} & GL_{26} & GL_{27} \\ GL_{42} & GL_{44} & GL_{45} & GL_{46} & GL_{47} \\ GL_{52} & GL_{54} & GL_{55} & GL_{56} & GL_{57} \\ GL_{62} & GL_{64} & GL_{65} & GL_{66} & GL_{67} \\ GL_{72} & GL_{74} & GL_{75} & GL_{76} & GL_{77} \end{pmatrix} \begin{pmatrix} U_2^0 \\ U_4^0 \\ U_5^0 \\ U_6^0 \\ U_7^0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{1}$$

With the material parameters listed in table 1, equations (1) must be rewritten as follows:

$$\begin{pmatrix} C\left[m-\left(V_{ph}/V_{t4}\right)^2\right] & em & hm & gm & fm \\ em & -\varepsilon m & -\alpha m & -\zeta m & -\xi m \\ hm & -\alpha m & -\mu m & -\beta m & -\lambda m \\ gm & -\zeta m & -\beta m & -\gamma m & -\vartheta m \\ fm & -\xi m & -\lambda m & -\vartheta m & -\eta m \end{pmatrix} \begin{pmatrix} U^0 \\ \phi^0 \\ \psi^0 \\ \Phi^0 \\ \Psi^0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{2}$$

where

$$m = 1 + n_3^2 \tag{3}$$

$$n_3 = k_3/k \tag{4}$$

$$(U^0, \phi^0, \psi^0, \Phi^0, \Psi^0) = (U_2^0, U_4^0, U_5^0, U_6^0, U_7^0) \tag{5}$$

$$V_{t4} = \sqrt{C/\rho} \tag{6}$$

$$V_{ph} = \omega / k \tag{7}$$

All the suitable eigenvalues n_3 and the corresponding eigenvectors respectively defined by expressions (4) and (5) must be obtained by resolving equation (2). The velocity V_{t4} defined by expression (6) represents the speed of the purely mechanical bulk SH-wave. The phase velocity V_{ph} defined by expression (7) is proportional to the angular frequency ω and inversely proportional to the wavenumber k in the propagation direction.

Expanding the determinant of the coefficient matrix in equations (2), it is possible to obtain a secular equation representing a polynomial that must be equal to zero. To find all the possible polynomial roots n_3, it is convenient to rewrite the polynomial in the form of several suitable cofactors in order that each cofactor contains two polynomial roots n_3. Therefore, it is possible to write down the following convenient form consisting of five cofactors:

$$m \times m \times m \times m \times \det(V_{ph}) = 0 \tag{8}$$

where

$$\det(V_{ph}) = \begin{vmatrix} C\left[m - \left(V_{ph}/V_{t4}\right)^2\right] & e & h & g & f \\ em & -\varepsilon & -\alpha & -\zeta & -\xi \\ hm & -\alpha & -\mu & -\beta & -\lambda \\ gm & -\zeta & -\beta & -\gamma & -\vartheta \\ fm & -\xi & -\lambda & -\vartheta & -\eta \end{vmatrix} \tag{9}$$

It is clearly seen that the first four cofactors in equation (8) are identical. Therefore, they actually release the following four pairs of identical eigenvalues:

$$n_3^{(1,2)} = n_3^{(3,4)} = n_3^{(5,6)} = n_3^{(7,8)} = \pm j \tag{10}$$

where $j = \sqrt{-1}$ is the imaginary unity.

Expanding determinant (9) leads to the following fifth pair of the eigenvalues:

$$n_3^{(9,10)} = \pm j\sqrt{1 - \left(V_{ph}/V_{temgc}\right)^2} \tag{11}$$

where

$$V_{temgc} = \sqrt{C/\rho}\left(1 + K_{emgc}^2\right)^{1/2} \tag{12}$$

$$K_{emgc}^2 = \frac{A_1}{A_2} \tag{13}$$

$$A_1 = e^2(\mu\gamma\eta + 2\beta\lambda\vartheta - \lambda^2\gamma - \beta^2\eta - \vartheta^2\mu) + h^2(\varepsilon\gamma\eta + 2\zeta\xi\vartheta - \vartheta^2\varepsilon - \zeta^2\eta - \xi^2\gamma) + g^2(\varepsilon\mu\eta + 2\alpha\xi\lambda - \lambda^2\varepsilon - \alpha^2\eta - \xi^2\mu) + f^2(\varepsilon\mu\gamma + 2\alpha\beta\zeta - \beta^2\varepsilon - \alpha^2\gamma - \zeta^2\mu) + 2eh(\vartheta^2\alpha + \zeta\beta\eta + \xi\gamma\lambda - \alpha\gamma\eta - \zeta\lambda\vartheta - \xi\beta\vartheta) + 2eg(\alpha\beta\eta + \lambda^2\zeta + \xi\vartheta\mu - \alpha\lambda\vartheta - \zeta\mu\eta - \xi\beta\lambda) + 2ef(\alpha\gamma\lambda + \zeta\vartheta\mu + \beta^2\xi - \alpha\beta\vartheta - \zeta\beta\lambda - \xi\mu\gamma) + 2hg(\varepsilon\lambda\vartheta + \zeta\alpha\eta + \xi^2\beta - \varepsilon\eta\beta - \zeta\lambda\xi - \xi\vartheta\alpha) + 2hf(\varepsilon\beta\vartheta + \zeta^2\lambda + \xi\alpha\gamma - \varepsilon\lambda\gamma - \zeta\vartheta\alpha - \xi\zeta\beta) + 2gf(\varepsilon\beta\lambda + \alpha^2\vartheta + \xi\mu\zeta - \varepsilon\mu\vartheta - \alpha\zeta\lambda - \alpha\beta\xi) \tag{14}$$

$$A_2 = C(\varepsilon\mu - \alpha^2)(\gamma\eta - \vartheta^2) + C(\beta^2\xi^2 - \xi^2\mu\gamma - \beta^2\varepsilon\eta) + C(\lambda^2\zeta^2 - \lambda^2\varepsilon\gamma - \zeta^2\mu\eta) + 2C(\gamma\alpha\xi\lambda + \eta\alpha\beta\zeta + \varepsilon\beta\lambda\vartheta + \mu\zeta\xi\vartheta - \zeta\xi\beta\lambda - \alpha\zeta\lambda\vartheta - \alpha\beta\xi\vartheta) \tag{15}$$

Table 2: The physical dimensions of factors M_{ee}, M_{hh}, M_{gg}, M_{ff}, M_{eh}, M_{eg}, M_{ef}, M_{hg}, M_{hf}, M_{gf} and CEMGCMC coupling mechanisms M_1, M_2, M_3, M_4, M_5.

Factor	Dimension
M_{ee}	s^2/m^2
M_{hh}	s^4/m^4
M_{gg}	$s^2/(kg\times m)$
M_{ff}	$kg\times s^4/m^5$
M_{eh}	s^3/m^3
M_{eg}	$s^2/(kg^{1/2}\times m^{3/2})$
M_{ef}	$kg^{1/2}\times s^3/m^{7/2}$
M_{hg}	$s^3/(kg^{1/2}\times m^{5/2})$
M_{hf}	$kg^{1/2}\times s^4/m^{9/2}$
M_{gf}	s^3/m^3
M_1	$kg^{1/2}\times s^2/m^{7/2}$
M_2	$kg^{1/2}\times s^3/m^{9/2}$
M_3	s^2/m^3
M_4	$kg\times s^3/m^5$
M_5	s^4/m^4
eM_1, hM_2, gM_3, fM_4, CM_5	$kg\times s^2/m^5$

In definition (12), the velocity V_{temgc} of the shear-horizontal bulk acoustic wave (SH-BAW) is coupled with the electrical φ, magnetic ψ, gravitational Φ, and cogravitational Ψ potentials. This velocity depends on the nondimensional parameter K^2_{emgc} (13) called the coefficient of the electromagnetogravitocogravitomechanical coupling (CEMGCMC). It is useful to rewrite the parameter K^2_{emgc} (13) as a function of the CEMGCMC coupling mechanisms M_1, M_2, M_3, M_4, and M_5. Their physical dimensions are listed in table 2. So, the parameter K^2_{emgc} reads:

$$K^2_{emgc} = \frac{eM_1+hM_2+gM_3+fM_4}{CM_5} \tag{16}$$

where

$$M_1 = eM_{ee} + hM_{eh} + gM_{eg} + fM_{ef} \tag{17}$$

$$M_2 = eM_{eh} + hM_{hh} + gM_{hg} + fM_{hf} \tag{18}$$

$$M_3 = eM_{eg} + hM_{hg} + gM_{gg} + fM_{gf} \tag{19}$$

$$M_4 = eM_{ef} + hM_{hf} + gM_{gf} + fM_{ff} \tag{20}$$

$$\begin{aligned} M_5 = \varepsilon M_{ee} + \alpha M_{eh} + \zeta M_{eg} + \xi M_{ef} = \alpha M_{eh} + \mu M_{hh} + \beta M_{hg} + \lambda M_{hf} = \\ \zeta M_{eg} + \beta M_{hg} + \gamma M_{gg} + \vartheta M_{gf} = \xi M_{ef} + \lambda M_{hf} + \vartheta M_{gf} + \eta M_{ff} = \\ (\varepsilon\mu - \alpha^2)(\gamma\eta - \vartheta^2) + (\beta\xi - \lambda\zeta)^2 - (\xi^2\mu\gamma + \beta^2\varepsilon\eta + \lambda^2\varepsilon\gamma + \zeta^2\mu\eta) + \\ 2(\gamma\alpha\xi\lambda + \eta\alpha\beta\zeta + \varepsilon\beta\lambda\vartheta + \mu\zeta\xi\vartheta - \alpha\zeta\lambda\vartheta - \alpha\beta\xi\vartheta) \end{aligned} \tag{21}$$

because

$$M_{ee} = \mu\gamma\eta + 2\beta\lambda\vartheta - \lambda^2\gamma - \beta^2\eta - \vartheta^2\mu \tag{22}$$

$$M_{hh} = \varepsilon\gamma\eta + 2\zeta\xi\vartheta - \vartheta^2\varepsilon - \zeta^2\eta - \xi^2\gamma \tag{23}$$

$$M_{gg} = \varepsilon\mu\eta + 2\alpha\xi\lambda - \lambda^2\varepsilon - \alpha^2\eta - \xi^2\mu \tag{24}$$

$$M_{ff} = \varepsilon\mu\gamma + 2\alpha\beta\zeta - \beta^2\varepsilon - \alpha^2\gamma - \zeta^2\mu \tag{25}$$

$$M_{eh} = \zeta\beta\eta + \xi\gamma\lambda + \vartheta^2\alpha - \alpha\gamma\eta - \zeta\lambda\vartheta - \xi\beta\vartheta \tag{26}$$

$$M_{eg} = \alpha\beta\eta + \xi\vartheta\mu + \lambda^2\zeta - \alpha\lambda\vartheta - \zeta\mu\eta - \xi\beta\lambda \tag{27}$$

$$M_{ef} = \alpha\gamma\lambda + \zeta\vartheta\mu + \beta^2\xi - \alpha\beta\vartheta - \zeta\beta\lambda - \xi\mu\gamma \tag{28}$$

$$M_{hg} = \varepsilon\lambda\vartheta + \zeta\alpha\eta + \xi^2\beta - \alpha\xi\vartheta - \zeta\lambda\xi - \varepsilon\eta\beta \tag{29}$$

$$M_{hf} = \varepsilon\beta\vartheta + \xi\alpha\gamma + \zeta^2\lambda - \alpha\zeta\vartheta - \zeta\xi\beta - \varepsilon\lambda\gamma \tag{30}$$

$$M_{gf} = \varepsilon\beta\lambda + \xi\mu\zeta + \alpha^2\vartheta - \alpha\zeta\lambda - \alpha\beta\xi - \varepsilon\mu\vartheta \tag{31}$$

The physical dimensions of parameters (22)-(31) are listed in table 2. To obtain the eigenvector components, it is necessary to resolve equation (2). This was done in paper [24]. However, paper [24] has demonstrated only half of twenty-four possible eigenvectors. Moreover, each of the twenty-four eigenvectors can be reduced to one of four main independent eigenvectors that will be obtained together with their properties in the next four sections. These independent eigenvectors are called the *F*-eigenvectors, *G*-eigenvectors, *H*-eigenvectors, and *E*-eigenvectors. In conclusion to this section, it is useful to write down some useful expressions for the nondimensional material parameters defined below by equalities from (32) to (41). These material parameters are used in the following four sections. For them, it is necessary to mention that the coefficient K^2_{emgc} (CEMGCMC) is defined by expression (16) and all the material constants are listed in table 1. These nondimensional material parameters read:

$$K_E = 1 - \frac{K_e^2}{K^2_{emgc}}, K_e^2 = \frac{e^2}{C\varepsilon} \tag{32}$$

$$K_M = 1 - \frac{K_m^2}{K^2_{emgc}}, K_m^2 = \frac{h^2}{C\mu} \tag{33}$$

$$K_G = 1 - \frac{K_g^2}{K^2_{emgc}}, K_g^2 = \frac{g^2}{C\gamma} \tag{34}$$

$$K_F = 1 - \frac{K_f^2}{K_{emgc}^2}, K_f^2 = \frac{f^2}{C\eta} \tag{35}$$

$$K_A = 1 - \frac{K_\alpha^2}{K_{emgc}^2}, K_\alpha^2 = \frac{eh}{C\alpha} \tag{36}$$

$$K_T = 1 - \frac{K_\vartheta^2}{K_{emgc}^2}, K_\vartheta^2 = \frac{gf}{C\vartheta} \tag{37}$$

$$K_B = 1 - \frac{K_\beta^2}{K_{emgc}^2}, K_\beta^2 = \frac{hg}{C\beta} \tag{38}$$

$$K_Z = 1 - \frac{K_\zeta^2}{K_{emgc}^2}, K_\zeta^2 = \frac{eg}{C\zeta} \tag{39}$$

$$K_S = 1 - \frac{K_\xi^2}{K_{emgc}^2}, K_\xi^2 = \frac{ef}{C\xi} \tag{40}$$

$$K_L = 1 - \frac{K_\lambda^2}{K_{emgc}^2}, K_\lambda^2 = \frac{hf}{C\lambda} \tag{41}$$

3 The *F*-eigenvectors and their properties

The first pair of the eigenvectors called the first and second *F*-eigenvectors is defined below by expressions (42) and (43), respectively. Eigenvector (42) relates to eigenvalues (10) and eigenvector (43) relates to eigenvalues (11). For eigenvector (42), the eigenvector components correspondingly depend on the parameters M_{ef}, M_{hf}, M_{gf}, and M_{ff} that are defined by expressions (28), (30), (31), and (25), respectively, and contain the subscript "*f*" symbolizing the *F*-eigenvectors. So, the *F*-eigenvectors can be written in the following convenient forms:

$$\begin{pmatrix} U^{0(1)} \\ \phi^{0(1)} \\ \psi^{0(1)} \\ \Phi^{0(1)} \\ \Psi^{0(1)} \end{pmatrix} = \begin{pmatrix} U^{0(2)} \\ \phi^{0(2)} \\ \psi^{0(2)} \\ \Phi^{0(2)} \\ \Psi^{0(2)} \end{pmatrix} = \begin{pmatrix} U^{0(3)} \\ \phi^{0(3)} \\ \psi^{0(3)} \\ \Phi^{0(3)} \\ \Psi^{0(3)} \end{pmatrix} = \begin{pmatrix} U^{0(4)} \\ \phi^{0(4)} \\ \psi^{0(4)} \\ \Phi^{0(4)} \\ \Psi^{0(4)} \end{pmatrix} = \begin{pmatrix} U^{0(5)} \\ \phi^{0(5)} \\ \psi^{0(5)} \\ \Phi^{0(5)} \\ \Psi^{0(5)} \end{pmatrix} = \begin{pmatrix} U^{0(6)} \\ \phi^{0(6)} \\ \psi^{0(6)} \\ \Phi^{0(6)} \\ \Psi^{0(6)} \end{pmatrix} =$$

$$\begin{pmatrix} U^{0(7)} \\ \phi^{0(7)} \\ \psi^{0(7)} \\ \Phi^{0(7)} \\ \Psi^{0(7)} \end{pmatrix} = \begin{pmatrix} U^{0(8)} \\ \phi^{0(8)} \\ \psi^{0(8)} \\ \Phi^{0(8)} \\ \Psi^{0(8)} \end{pmatrix} = \begin{pmatrix} U_f^{0(7)} = 0 \\ \phi_f^{0(7)} = M_{ef} \\ \psi_f^{0(7)} = M_{hf} \\ \Phi_f^{0(7)} = M_{gf} \\ \Psi_f^{0(7)} = M_{ff} \end{pmatrix} \tag{42}$$

$$\begin{pmatrix} U^{0(9)} \\ \phi^{0(9)} \\ \psi^{0(9)} \\ \Phi^{0(9)} \\ \Psi^{0(9)} \end{pmatrix} = \begin{pmatrix} U^{0(10)} \\ \phi^{0(10)} \\ \psi^{0(10)} \\ \Phi^{0(10)} \\ \Psi^{0(10)} \end{pmatrix} = \begin{pmatrix} U_f^{0(9)} \\ \phi_f^{0(9)} \\ \psi_f^{0(9)} \\ \Phi_f^{0(9)} \\ \Psi_f^{0(9)} \end{pmatrix} \tag{43}$$

where

$$U_f^{0(9)} = \frac{e\phi_f^{0(9)} + h\psi_f^{0(9)} + g\Phi_f^{0(9)} + f\Psi_f^{0(9)}}{CK_{emgc}^2} = \frac{M_4}{CK_{emgc}^2} \tag{44}$$

$$\phi_f^{0(9)} = \gamma\alpha\lambda K_G K_A K_L + \mu\vartheta\zeta K_M K_T K_Z + \beta^2\xi K_B{}^2 K_S - \alpha\beta\vartheta K_A K_B K_T - \beta\lambda\zeta K_B K_L K_Z - \mu\gamma\xi K_M K_G K_S = \phi_f^{0(7)} - \gamma\alpha\lambda\left(\frac{K_g^2}{K_{emgc}^2} + \frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2}\right) - \mu\vartheta\zeta\left(\frac{K_m^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) - \beta^2\xi\left(2\frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) + \alpha\beta\vartheta\left(\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2}\right) + \beta\lambda\zeta\left(\frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) + \mu\gamma\xi\left(\frac{K_m^2}{K_{emgc}^2} + \frac{K_g^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) \tag{45}$$

$$\psi_f^{0(9)} = \gamma\alpha\xi K_G K_A K_S + \lambda\zeta^2 K_L {K_Z}^2 + \varepsilon\beta\vartheta K_E K_B K_T - \alpha\vartheta\zeta K_A K_T K_Z - \beta\xi\zeta K_B K_S K_Z - \varepsilon\gamma\lambda K_E K_G K_L = \psi_f^{0(7)} - \gamma\alpha\xi\left(\frac{K_g^2}{K_{emgc}^2} + \frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) - \lambda\zeta^2\left(\frac{K_\lambda^2}{K_{emgc}^2} + 2\frac{K_\zeta^2}{K_{emgc}^2}\right) - \varepsilon\beta\vartheta\left(\frac{K_e^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2}\right) + \alpha\vartheta\zeta\left(\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) + \beta\xi\zeta\left(\frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) + \varepsilon\gamma\lambda\left(\frac{K_e^2}{K_{emgc}^2} + \frac{K_g^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2}\right) \quad (46)$$

$$\Phi_f^{0(9)} = \alpha^2\vartheta {K_A}^2 K_T + \mu\xi\zeta K_M K_S K_Z + \varepsilon\beta\lambda K_E K_B K_L - \alpha\lambda\zeta K_A K_L K_Z - \alpha\beta\xi K_A K_B K_S - \varepsilon\mu\vartheta K_E K_M K_T = \Phi_f^{0(7)} - \alpha^2\vartheta\left(2\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2}\right) - \mu\xi\zeta\left(\frac{K_m^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) - \varepsilon\beta\lambda\left(\frac{K_e^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2}\right) + \alpha\lambda\zeta\left(\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) + \alpha\beta\xi\left(\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) + \varepsilon\mu\vartheta\left(\frac{K_e^2}{K_{emgc}^2} + \frac{K_m^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2}\right) \quad (47)$$

$$\Psi_f^{0(9)} = 2\alpha\beta\zeta K_A K_B K_Z + \varepsilon\mu\gamma K_E K_M K_G - \gamma\alpha^2 K_G {K_A}^2 - \mu\zeta^2 K_M {K_Z}^2 - \varepsilon\beta^2 K_E {K_B}^2 = \Psi_f^{0(7)} - 2\alpha\beta\zeta\left(\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) - \varepsilon\mu\gamma\left(\frac{K_e^2}{K_{emgc}^2} + \frac{K_m^2}{K_{emgc}^2} + \frac{K_g^2}{K_{emgc}^2}\right) + \gamma\alpha^2\left(\frac{K_g^2}{K_{emgc}^2} + 2\frac{K_\alpha^2}{K_{emgc}^2}\right) + \mu\zeta^2\left(\frac{K_m^2}{K_{emgc}^2} + 2\frac{K_\zeta^2}{K_{emgc}^2}\right) + \varepsilon\beta^2\left(\frac{K_e^2}{K_{emgc}^2} + 2\frac{K_\beta^2}{K_{emgc}^2}\right) \quad (48)$$

It is clearly seen in expressions (45), (46), (47), and (48) that the eigenvector components $\phi_f^{0(9)}$, $\psi_f^{0(9)}$, $\Phi_f^{0(9)}$, and $\Psi_f^{0(9)}$ depend on the corresponding eigenvector components $\phi_f^{0(7)}$, $\psi_f^{0(7)}$, $\Phi_f^{0(7)}$, and $\Psi_f^{0(7)}$. This manifests that both eigenvectors (42) and (43) are naturally called the *F*-eigenvectors. The analogical reason was used for distinguishing of the rest three independent

eigenvectors called the *G*-, *H*-, and *E*-eigenvectors in the following three sections.

With the CEMGCMC coupling mechanisms M_4 and M_5 defined by expressions (20) and (21), respectively, the properties of the corresponding components of the *F*-eigenvectors can be demonstrated by all the expressions written below in this section. For the linear combinations of the components of the first *F*-eigenvector, the following five expressions disclose some useful properties:

$$e\phi_f^{0(7)} + h\psi_f^{0(7)} + g\Phi_f^{0(7)} + f\Psi_f^{0(7)} = e\phi_f^{0(9)} + h\psi_f^{0(9)} + g\Phi_f^{0(9)} + f\Psi_f^{0(9)} = eM_{ef} + hM_{hf} + gM_{gf} + fM_{ff} = M_4 \quad (49)$$

$$\varepsilon\phi_f^{0(7)} + \alpha\psi_f^{0(7)} + \zeta\Phi_f^{0(7)} + \xi\Psi_f^{0(7)} = \varepsilon K_E\phi_f^{0(9)} + \alpha K_A\psi_f^{0(9)} + \zeta K_Z\Phi_f^{0(9)} + \xi K_S\Psi_f^{0(9)} = 0 \quad (50)$$

$$\alpha\phi_f^{0(7)} + \mu\psi_f^{0(7)} + \beta\Phi_f^{0(7)} + \lambda\Psi_f^{0(7)} = \alpha K_A\phi_f^{0(9)} + \mu K_M\psi_f^{0(9)} + \beta K_B\Phi_f^{0(9)} + \lambda K_L\Psi_f^{0(9)} = 0 \quad (51)$$

$$\zeta\phi_f^{0(7)} + \beta\psi_f^{0(7)} + \gamma\Phi_f^{0(7)} + \vartheta\Psi_f^{0(7)} = \zeta K_Z\phi_f^{0(9)} + \beta K_B\psi_f^{0(9)} + \gamma K_G\Phi_f^{0(9)} + \vartheta K_T\Psi_f^{0(9)} = 0 \quad (52)$$

$$\xi\phi_f^{0(7)} + \lambda\psi_f^{0(7)} + \vartheta\Phi_f^{0(7)} + \eta\Psi_f^{0(7)} = \xi M_{ef} + \lambda M_{hf} + \vartheta M_{gf} + \eta M_{ff} = M_5 \quad (53)$$

For the linear combinations of the components of the second *F*-eigenvector, the following equalities can be obtained:

$$\varepsilon\phi_f^{0(9)} + \alpha\psi_f^{0(9)} + \zeta\Phi_f^{0(9)} + \xi\Psi_f^{0(9)} = -\varepsilon K_E\phi_f^{0(7)} - \alpha K_A\psi_f^{0(7)} - \zeta K_Z\Phi_f^{0(7)} - \xi K_S\Psi_f^{0(7)} = \frac{e}{CK_{emgc}^2}\left(e\phi_f^{0(7)} + h\psi_f^{0(7)} + g\Phi_f^{0(7)} + f\Psi_f^{0(7)}\right) = \frac{eM_4}{CK_{emgc}^2} \quad (54)$$

$$\alpha\phi_f^{0(9)} + \mu\psi_f^{0(9)} + \beta\Phi_f^{0(9)} + \lambda\Psi_f^{0(9)} = -\alpha K_A\phi_f^{0(7)} - \mu K_M\psi_f^{0(7)} - \beta K_B\Phi_f^{0(7)} - \lambda K_L\Psi_f^{0(7)} = \frac{h}{CK_{emgc}^2}\left(e\phi_f^{0(7)} + h\psi_f^{0(7)} + g\Phi_f^{0(7)} + f\Psi_f^{0(7)}\right) = \frac{hM_4}{CK_{emgc}^2} \quad (55)$$

$$\zeta\phi_f^{0(9)} + \beta\psi_f^{0(9)} + \gamma\Phi_f^{0(9)} + \vartheta\dot{\Psi}_f^{0(9)} = -\zeta K_Z\phi_f^{0(7)} - \beta K_B\psi_f^{0(7)} - \gamma K_G\Phi_f^{0(7)} - \vartheta K_T\Psi_f^{0(7)} = \frac{g}{CK_{emgc}^2}\left(e\phi_f^{0(7)} + h\psi_f^{0(7)} + g\Phi_f^{0(7)} + f\Psi_f^{0(7)}\right) = \frac{gM_4}{CK_{emgc}^2} \quad (56)$$

$$\xi K_S\phi_f^{0(9)} + \lambda K_L\psi_f^{0(9)} + \vartheta K_T\Phi_f^{0(9)} + \eta K_F\Psi_f^{0(9)} = 0 \quad (57)$$

Using property (57), it is possible to write down the following equalities:

$$\xi\phi_f^{0(9)} + \lambda\psi_f^{0(9)} + \vartheta\Phi_f^{0(9)} + \eta\Psi_f^{0(9)} = \xi\frac{K_\xi^2}{K_{emgc}^2}\phi_f^{0(9)} + \lambda\frac{K_\lambda^2}{K_{emgc}^2}\psi_f^{0(9)} + \vartheta\frac{K_\vartheta^2}{K_{emgc}^2}\Phi_f^{0(9)} + \eta\frac{K_f^2}{K_{emgc}^2}\Psi_f^{0(9)} = \frac{f}{CK_{emgc}^2}\left(e\phi_f^{0(9)} + h\psi_f^{0(9)} + g\Phi_f^{0(9)} + f\Psi_f^{0(9)}\right) = \frac{f}{CK_{emgc}^2}\left(e\phi_f^{0(7)} + h\psi_f^{0(7)} + g\Phi_f^{0(7)} + f\Psi_f^{0(7)}\right) = \xi\frac{K_\xi^2}{K_{emgc}^2}\phi_f^{0(7)} + \lambda\frac{K_\lambda^2}{K_{emgc}^2}\psi_f^{0(7)} + \vartheta\frac{K_\vartheta^2}{K_{emgc}^2}\Phi_f^{0(7)} + \eta\frac{K_f^2}{K_{emgc}^2}\Psi_f^{0(7)} = \frac{fM_4}{CK_{emgc}^2} \quad (58)$$

4 The *G*-eigenvectors and their properties

The explicit forms of the second pair of the independent eigenvectors called the *G*-eigenvectors are obtained below in expressions (59) and (60). For first *G*-eigenvector (59), the eigenvector components correspondingly depend on the parameters M_{eg}, M_{hg}, M_{gg}, and M_{gf} that are defined by expressions (27), (29), (24), and (31), respectively, and contain the subscript "*g*" symbolizing the *G*-eigenvectors. So, the first and second *G*-eigenvectors respectively read:

$$\begin{pmatrix} U^{0(1)} \\ \phi^{0(1)} \\ \psi^{0(1)} \\ \Phi^{0(1)} \\ \Psi^{0(1)} \end{pmatrix} = \begin{pmatrix} U^{0(2)} \\ \phi^{0(2)} \\ \psi^{0(2)} \\ \Phi^{0(2)} \\ \Psi^{0(2)} \end{pmatrix} = \begin{pmatrix} U^{0(3)} \\ \phi^{0(3)} \\ \psi^{0(3)} \\ \Phi^{0(3)} \\ \Psi^{0(3)} \end{pmatrix} = \begin{pmatrix} U^{0(4)} \\ \phi^{0(4)} \\ \psi^{0(4)} \\ \Phi^{0(4)} \\ \Psi^{0(4)} \end{pmatrix} = \begin{pmatrix} U^{0(5)} \\ \phi^{0(5)} \\ \psi^{0(5)} \\ \Phi^{0(5)} \\ \Psi^{0(5)} \end{pmatrix} = \begin{pmatrix} U^{0(6)} \\ \phi^{0(6)} \\ \psi^{0(6)} \\ \Phi^{0(6)} \\ \Psi^{0(6)} \end{pmatrix} =$$

$$\begin{pmatrix} U^{0(7)} \\ \phi^{0(7)} \\ \psi^{0(7)} \\ \Phi^{0(7)} \\ \Psi^{0(7)} \end{pmatrix} = \begin{pmatrix} U^{0(8)} \\ \phi^{0(8)} \\ \psi^{0(8)} \\ \Phi^{0(8)} \\ \Psi^{0(8)} \end{pmatrix} = \begin{pmatrix} U_g^{0(7)} = 0 \\ \phi_g^{0(7)} = M_{eg} \\ \psi_g^{0(7)} = M_{hg} \\ \Phi_g^{0(7)} = M_{gg} \\ \Psi_g^{0(7)} = M_{gf} \end{pmatrix} \tag{59}$$

$$\begin{pmatrix} U^{0(9)} \\ \phi^{0(9)} \\ \psi^{0(9)} \\ \Phi^{0(9)} \\ \Psi^{0(9)} \end{pmatrix} = \begin{pmatrix} U^{0(10)} \\ \phi^{0(10)} \\ \psi^{0(10)} \\ \Phi^{0(10)} \\ \Psi^{0(10)} \end{pmatrix} = \begin{pmatrix} U_g^{0(9)} \\ \phi_g^{0(9)} \\ \psi_g^{0(9)} \\ \Phi_g^{0(9)} \\ \Psi_g^{0(9)} \end{pmatrix} \tag{60}$$

where

$$U_g^{0(9)} = \frac{e\phi_g^{0(9)} + h\psi_g^{0(9)} + g\Phi_g^{0(9)} + f\Psi_g^{0(9)}}{CK_{emgc}^2} = \frac{M_3}{CK_{emgc}^2} \tag{61}$$

$$\phi_g^{0(9)} = \eta\alpha\beta K_F K_A K_B + \mu\vartheta\xi K_M K_T K_S + \lambda^2\zeta {K_L}^2 K_Z - \alpha\vartheta\lambda K_A K_T K_L - \mu\eta\zeta K_M K_F K_Z - \beta\lambda\xi K_B K_L K_S = \phi_g^{0(7)} - \eta\alpha\beta\left(\frac{K_f^2}{K_{emgc}^2} + \frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2}\right) - \mu\vartheta\xi\left(\frac{K_m^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) - \lambda^2\zeta\left(2\frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) + \alpha\vartheta\lambda\left(\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2}\right) + \mu\eta\zeta\left(\frac{K_m^2}{K_{emgc}^2} + \frac{K_f^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) + \beta\lambda\xi\left(\frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) \tag{62}$$

$$\psi_g^{0(9)} = \eta\alpha\zeta K_F K_A K_Z + \beta\xi^2 K_B K_S{}^2 + \varepsilon\vartheta\lambda K_E K_T K_L - \alpha\vartheta\xi K_A K_T K_S - \lambda\xi\zeta K_L K_S K_Z - \varepsilon\eta\beta K_E K_F K_B = \psi_g^{0(7)} - \eta\alpha\zeta\left(\frac{K_f^2}{K_{emgc}^2} + \frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) - \beta\xi^2\left(\frac{K_\beta^2}{K_{emgc}^2} + 2\frac{K_\xi^2}{K_{emgc}^2}\right) - \varepsilon\vartheta\lambda\left(\frac{K_e^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2}\right) + \alpha\vartheta\xi\left(\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) + \lambda\xi\zeta\left(\frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) + \varepsilon\eta\beta\left(\frac{K_e^2}{K_{emgc}^2} + \frac{K_f^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2}\right) \quad (63)$$

$$\Phi_g^{0(9)} = 2\alpha\lambda\xi K_A K_L K_S + \varepsilon\mu\eta K_E K_M K_F - \eta\alpha^2 K_F K_A{}^2 - \mu\xi^2 K_M K_S{}^2 - \varepsilon\lambda^2 K_E K_L{}^2 = \Phi_g^{0(7)} - 2\alpha\lambda\xi\left(\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) - \varepsilon\mu\eta\left(\frac{K_e^2}{K_{emgc}^2} + \frac{K_m^2}{K_{emgc}^2} + \frac{K_f^2}{K_{emgc}^2}\right) + \eta\alpha^2\left(\frac{K_f^2}{K_{emgc}^2} + 2\frac{K_\alpha^2}{K_{emgc}^2}\right) + \mu\xi^2\left(\frac{K_m^2}{K_{emgc}^2} + 2\frac{K_\xi^2}{K_{emgc}^2}\right) + \varepsilon\lambda^2\left(\frac{K_e^2}{K_{emgc}^2} + 2\frac{K_\lambda^2}{K_{emgc}^2}\right) \quad (64)$$

$$\Psi_g^{0(9)} = \alpha^2\vartheta K_A{}^2 K_T + \mu\xi\zeta K_M K_S K_Z + \varepsilon\beta\lambda K_E K_B K_L - \alpha\lambda\zeta K_A K_L K_Z - \alpha\beta\xi K_A K_B K_S - \varepsilon\mu\vartheta K_E K_M K_T = \Psi_g^{0(7)} - \alpha^2\vartheta\left(2\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2}\right) - \mu\xi\zeta\left(\frac{K_m^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) - \varepsilon\beta\lambda\left(\frac{K_e^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2}\right) + \alpha\lambda\zeta\left(\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) + \alpha\beta\xi\left(\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) + \varepsilon\mu\vartheta\left(\frac{K_e^2}{K_{emgc}^2} + \frac{K_m^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2}\right) \quad (65)$$

The properties of the corresponding *G*-eigenvectors' components can be illuminated by all the equalities written below in this section, where the CEMGCMC coupling mechanism M_3 is defined by expression (19). For the natural linear combinations of the components of the first *G*-eigenvector, the reader can find the following five expressions:

$$e\phi_g^{0(7)} + h\psi_g^{0(7)} + g\Phi_g^{0(7)} + f\Psi_g^{0(7)} = e\phi_g^{0(9)} + h\psi_g^{0(9)} + g\Phi_g^{0(9)} + f\Psi_g^{0(9)} = eM_{eg} + hM_{hg} + gM_{gg} + fM_{gf} = M_3 \quad (66)$$

$$\varepsilon\phi_g^{0(7)} + \alpha\psi_g^{0(7)} + \zeta\Phi_g^{0(7)} + \xi\Psi_g^{0(7)} = \varepsilon K_E\phi_g^{0(9)} + \alpha K_A\psi_g^{0(9)} + \zeta K_Z\Phi_g^{0(9)} + \xi K_S\Psi_g^{0(9)} = 0 \quad (67)$$

$$\alpha\phi_g^{0(7)} + \mu\psi_g^{0(7)} + \beta\Phi_g^{0(7)} + \lambda\Psi_g^{0(7)} = \alpha K_A\phi_g^{0(9)} + \mu K_M\psi_g^{0(9)} + \beta K_B\Phi_g^{0(9)} + \lambda K_L\Psi_g^{0(9)} = 0 \quad (68)$$

$$\zeta\phi_g^{0(7)} + \beta\psi_g^{0(7)} + \gamma\Phi_g^{0(7)} + \vartheta\Psi_g^{0(7)} = \zeta M_{eg} + \beta M_{hg} + \gamma M_{gg} + \vartheta M_{gf} = M_5 \quad (69)$$

$$\xi\phi_g^{0(7)} + \lambda\psi_g^{0(7)} + \vartheta\Phi_g^{0(7)} + \eta\Psi_g^{0(7)} = \xi K_S\phi_g^{0(9)} + \lambda K_L\psi_g^{0(9)} + \vartheta K_T\Phi_g^{0(9)} + \eta K_F\Psi_g^{0(9)} = 0 \quad (70)$$

For the linear combinations of the components of the second G-eigenvector, it is natural to demonstrate the following obtained properties shown in the five expressions below:

$$\varepsilon\phi_g^{0(9)} + \alpha\psi_g^{0(9)} + \zeta\Phi_g^{0(9)} + \xi\Psi_g^{0(9)} = -\varepsilon K_E\phi_g^{0(7)} - \alpha K_A\psi_g^{0(7)} - \zeta K_Z\Phi_g^{0(7)} - \xi K_S\Psi_g^{0(7)} = \frac{e}{CK_{emgc}^2}\left(e\phi_g^{0(7)} + h\psi_g^{0(7)} + g\Phi_g^{0(7)} + f\Psi_g^{0(7)}\right) = \frac{eM_3}{CK_{emgc}^2} \quad (71)$$

$$\alpha\phi_g^{0(9)} + \mu\psi_g^{0(9)} + \beta\Phi_g^{0(9)} + \lambda\Psi_g^{0(9)} = -\alpha K_A\phi_g^{0(7)} - \mu K_M\psi_g^{0(7)} - \beta K_B\Phi_g^{0(7)} - \lambda K_L\Psi_g^{0(7)} = \frac{h}{CK_{emgc}^2}\left(e\phi_g^{0(7)} + h\psi_g^{0(7)} + g\Phi_g^{0(7)} + f\Psi_g^{0(7)}\right) = \frac{hM_3}{CK_{emgc}^2} \quad (72)$$

$$\zeta K_Z\phi_g^{0(9)} + \beta K_B\psi_g^{0(9)} + \gamma K_G\Phi_g^{0(9)} + \vartheta K_T\Psi_g^{0(9)} = 0 \quad (73)$$

Using equality (73), it is possible to obtain the following equalities:

$$\zeta\phi_g^{0(9)} + \beta\psi_g^{0(9)} + \gamma\Phi_g^{0(9)} + \vartheta\Psi_g^{0(9)} = \zeta\frac{K_\zeta^2}{K_{emgc}^2}\phi_g^{0(9)} + \beta\frac{K_\beta^2}{K_{emgc}^2}\psi_g^{0(9)} + \gamma\frac{K_g^2}{K_{emgc}^2}\Phi_g^{0(9)} + \vartheta\frac{K_\vartheta^2}{K_{emgc}^2}\Psi_g^{0(9)} = \frac{g}{CK_{emgc}^2}\left(e\phi_g^{0(9)} + h\psi_g^{0(9)} + g\Phi_g^{0(9)} +\right.$$

$$f\Psi_g^{0(9)}\Big) = \frac{g}{CK_{emgc}^2}\Big(e\phi_g^{0(7)} + h\psi_g^{0(7)} + g\Phi_g^{0(7)} + f\Psi_g^{0(7)}\Big) = \zeta\frac{K_\zeta^2}{K_{emgc}^2}\phi_g^{0(7)} + \beta\frac{K_\beta^2}{K_{emgc}^2}\psi_g^{0(7)} + \gamma\frac{K_g^2}{K_{emgc}^2}\Phi_g^{0(7)} + \vartheta\frac{K_\vartheta^2}{K_{emgc}^2}\Psi_g^{0(7)} = \frac{gM_3}{CK_{emgc}^2} \quad (74)$$

For the last linear combination, one can get the following equality:

$$\xi\phi_g^{0(9)} + \lambda\psi_g^{0(9)} + \vartheta\Phi_g^{0(9)} + \eta\Psi_g^{0(9)} = -\xi K_S\phi_g^{0(7)} - \lambda K_L\psi_g^{0(7)} - \vartheta K_T\Phi_g^{0(7)} - \eta K_F\Psi_g^{0(7)} = \frac{g}{CK_{emgc}^2}\Big(e\phi_g^{0(7)} + h\psi_g^{0(7)} + g\Phi_g^{0(7)} + f\Psi_g^{0(7)}\Big) = \frac{fM_3}{CK_{emgc}^2} \quad (75)$$

5 The *H*-eigenvectors and their properties

The third independent pair of the suitable eigenvectors is naturally called the *H*-eigenvectors. Their explicit forms are given below in expressions (76) and (77). For first eigenvector (76), the eigenvector components respectively represent the parameters M_{eh}, M_{hh}, M_{hg}, and M_{hf}. These parameters are defined by expressions (26), (23), (29), and (30), respectively, and contain the subscript "*h*" symbolizing the *H*-eigenvectors. Accordingly, the first and second *H*-eigenvectors are respectively written as follows:

$$\begin{pmatrix} U^{0(1)} \\ \phi^{0(1)} \\ \psi^{0(1)} \\ \Phi^{0(1)} \\ \Psi^{0(1)} \end{pmatrix} = \begin{pmatrix} U^{0(2)} \\ \phi^{0(2)} \\ \psi^{0(2)} \\ \Phi^{0(2)} \\ \Psi^{0(2)} \end{pmatrix} = \begin{pmatrix} U^{0(3)} \\ \phi^{0(3)} \\ \psi^{0(3)} \\ \Phi^{0(3)} \\ \Psi^{0(3)} \end{pmatrix} = \begin{pmatrix} U^{0(4)} \\ \phi^{0(4)} \\ \psi^{0(4)} \\ \Phi^{0(4)} \\ \Psi^{0(4)} \end{pmatrix} = \begin{pmatrix} U^{0(5)} \\ \phi^{0(5)} \\ \psi^{0(5)} \\ \Phi^{0(5)} \\ \Psi^{0(5)} \end{pmatrix} = \begin{pmatrix} U^{0(6)} \\ \phi^{0(6)} \\ \psi^{0(6)} \\ \Phi^{0(6)} \\ \Psi^{0(6)} \end{pmatrix} = \begin{pmatrix} U^{0(7)} \\ \phi^{0(7)} \\ \psi^{0(7)} \\ \Phi^{0(7)} \\ \Psi^{0(7)} \end{pmatrix} = \begin{pmatrix} U^{0(8)} \\ \phi^{0(8)} \\ \psi^{0(8)} \\ \Phi^{0(8)} \\ \Psi^{0(8)} \end{pmatrix} = \begin{pmatrix} U_h^{0(7)} = 0 \\ \phi_h^{0(7)} = M_{eh} \\ \psi_h^{0(7)} = M_{hh} \\ \Phi_h^{0(7)} = M_{hg} \\ \Psi_h^{0(7)} = M_{hf} \end{pmatrix} \quad (76)$$

$$\begin{pmatrix} U^{0(9)} \\ \phi^{0(9)} \\ \psi^{0(9)} \\ \Phi^{0(9)} \\ \Psi^{0(9)} \end{pmatrix} = \begin{pmatrix} U^{0(10)} \\ \phi^{0(10)} \\ \psi^{0(10)} \\ \Phi^{0(10)} \\ \Psi^{0(10)} \end{pmatrix} = \begin{pmatrix} U_h^{0(9)} \\ \phi_h^{0(9)} \\ \psi_h^{0(9)} \\ \Phi_h^{0(9)} \\ \Psi_h^{0(9)} \end{pmatrix} \tag{77}$$

where

$$U_h^{0(9)} = \frac{e\phi_h^{0(9)} + h\psi_h^{0(9)} + g\Phi_h^{0(9)} + f\Psi_h^{0(9)}}{CK_{emgc}^2} = \frac{M_2}{CK_{emgc}^2} \tag{78}$$

$$\phi_h^{0(9)} = \alpha\vartheta^2 K_A {K_T}^2 + \eta\beta\zeta K_F K_B K_Z + \gamma\lambda\xi K_G K_L K_S - \gamma\eta\alpha K_G K_F K_A - \vartheta\lambda\zeta K_T K_L K_Z - \beta\vartheta\xi K_B K_T K_S = \phi_h^{0(7)} - \alpha\vartheta^2\left(\frac{K_\alpha^2}{K_{emgc}^2} + 2\frac{K_\vartheta^2}{K_{emgc}^2}\right) - \eta\beta\zeta\left(\frac{K_f^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) - \gamma\lambda\xi\left(\frac{K_g^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) + \gamma\eta\alpha\left(\frac{K_g^2}{K_{emgc}^2} + \frac{K_f^2}{K_{emgc}^2} + \frac{K_\alpha^2}{K_{emgc}^2}\right) + \vartheta\lambda\zeta\left(\frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) + \beta\vartheta\xi\left(\frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) \tag{79}$$

$$\psi_h^{0(9)} = 2\vartheta\xi\zeta K_T K_S K_Z + \varepsilon\gamma\eta K_E K_G K_F - \gamma\xi^2 K_G {K_S}^2 - \eta\zeta^2 K_F {K_Z}^2 - \varepsilon\vartheta^2 K_E {K_T}^2 = \psi_h^{0(7)} - 2\vartheta\xi\zeta\left(\frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) - \varepsilon\gamma\eta\left(\frac{K_e^2}{K_{emgc}^2} + \frac{K_g^2}{K_{emgc}^2} + \frac{K_f^2}{K_{emgc}^2}\right) + \gamma\xi^2\left(\frac{K_g^2}{K_{emgc}^2} + 2\frac{K_\xi^2}{K_{emgc}^2}\right) + \eta\zeta^2\left(\frac{K_f^2}{K_{emgc}^2} + 2\frac{K_\zeta^2}{K_{emgc}^2}\right) + \varepsilon\vartheta^2\left(\frac{K_e^2}{K_{emgc}^2} + 2\frac{K_\vartheta^2}{K_{emgc}^2}\right) \tag{80}$$

$$\Phi_h^{0(9)} = \eta\alpha\zeta K_F K_A K_Z + \beta\xi^2 K_B {K_S}^2 + \varepsilon\vartheta\lambda K_E K_T K_L - \lambda\xi\zeta K_L K_S K_Z - \alpha\vartheta\xi K_A K_T K_S - \varepsilon\eta\beta K_E K_F K_B = \Phi_h^{0(7)} - \eta\alpha\zeta\left(\frac{K_f^2}{K_{emgc}^2} + \frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) -$$

$$\beta\xi^2\left(\frac{K_\beta^2}{K_{emgc}^2}+2\frac{K_\xi^2}{K_{emgc}^2}\right)-\varepsilon\vartheta\lambda\left(\frac{K_e^2}{K_{emgc}^2}+\frac{K_\vartheta^2}{K_{emgc}^2}+\frac{K_\lambda^2}{K_{emgc}^2}\right)+\lambda\xi\zeta\left(\frac{K_\lambda^2}{K_{emgc}^2}+\frac{K_\xi^2}{K_{emgc}^2}+\frac{K_\zeta^2}{K_{emgc}^2}\right)+\alpha\vartheta\xi\left(\frac{K_\alpha^2}{K_{emgc}^2}+\frac{K_\vartheta^2}{K_{emgc}^2}+\frac{K_\xi^2}{K_{emgc}^2}\right)+\varepsilon\eta\beta\left(\frac{K_e^2}{K_{emgc}^2}+\frac{K_f^2}{K_{emgc}^2}+\frac{K_\beta^2}{K_{emgc}^2}\right) \quad (81)$$

$$\Psi_h^{0(9)}=\gamma\alpha\xi K_G K_A K_S+\varepsilon\beta\vartheta K_E K_B K_T+\lambda\zeta^2 K_L K_Z{}^2-\alpha\vartheta\zeta K_A K_T K_Z-\beta\xi\zeta K_B K_S K_Z-\varepsilon\gamma\lambda K_E K_G K_L=\Psi_h^{0(7)}-\gamma\alpha\xi\left(\frac{K_g^2}{K_{emgc}^2}+\frac{K_\alpha^2}{K_{emgc}^2}+\frac{K_\xi^2}{K_{emgc}^2}\right)-\varepsilon\beta\vartheta\left(\frac{K_e^2}{K_{emgc}^2}+\frac{K_\beta^2}{K_{emgc}^2}+\frac{K_\vartheta^2}{K_{emgc}^2}\right)-\lambda\zeta^2\left(\frac{K_\lambda^2}{K_{emgc}^2}+2\frac{K_\zeta^2}{K_{emgc}^2}\right)+\alpha\vartheta\zeta\left(\frac{K_\alpha^2}{K_{emgc}^2}+\frac{K_\vartheta^2}{K_{emgc}^2}+\frac{K_\zeta^2}{K_{emgc}^2}\right)+\beta\xi\zeta\left(\frac{K_\beta^2}{K_{emgc}^2}+\frac{K_\xi^2}{K_{emgc}^2}+\frac{K_\zeta^2}{K_{emgc}^2}\right)+\varepsilon\gamma\lambda\left(\frac{K_e^2}{K_{emgc}^2}+\frac{K_g^2}{K_{emgc}^2}+\frac{K_\lambda^2}{K_{emgc}^2}\right) \quad (82)$$

The natural linear combinations of the corresponding H-eigenvectors' components disclose the properties obtained in all the expressions written below in this section, where the CEMGCMC coupling mechanisms M_2 and M_5 are defined by equations (18) and (21), respectively. For the linear combinations of the components of the first H-eigenvector, the following five properties can be obtained:

$$e\phi_h^{0(7)}+h\psi_h^{0(7)}+g\Phi_h^{0(7)}+f\Psi_h^{0(7)}=e\phi_h^{0(9)}+h\psi_h^{0(9)}+g\Phi_h^{0(9)}+f\Psi_h^{0(9)}=eM_{eh}+hM_{hh}+gM_{hg}+fM_{hf}=M_2 \quad (83)$$

$$\varepsilon\phi_h^{0(7)}+\alpha\psi_h^{0(7)}+\zeta\Phi_h^{0(7)}+\xi\Psi_h^{0(7)}=\varepsilon K_E\phi_h^{0(9)}+\alpha K_A\psi_h^{0(9)}+\zeta K_Z\Phi_h^{0(9)}+\xi K_S\Psi_h^{0(9)}=0 \quad (84)$$

$$\alpha\phi_h^{0(7)}+\mu\psi_h^{0(7)}+\beta\Phi_h^{0(7)}+\lambda\Psi_h^{0(7)}=\alpha M_{eh}+\mu M_{hh}+\beta M_{hg}+\lambda M_{hf}=M_5 \quad (85)$$

$$\zeta\phi_h^{0(7)}+\beta\psi_h^{0(7)}+\gamma\Phi_h^{0(7)}+\vartheta\Psi_h^{0(7)}=\zeta K_Z\phi_h^{0(9)}+\beta K_B\psi_h^{0(9)}+\gamma K_G\Phi_h^{0(9)}+\vartheta K_T\Psi_h^{0(9)}=0 \quad (86)$$

$$\xi\phi_h^{0(7)}+\lambda\psi_h^{0(7)}+\vartheta\Phi_h^{0(7)}+\eta\Psi_h^{0(7)}=\xi K_S\phi_h^{0(9)}+\lambda K_L\psi_h^{0(9)}+\vartheta K_T\Phi_h^{0(9)}+\eta K_F\Psi_h^{0(9)}=0 \quad (87)$$

For the linear combinations of the components of the second H-eigenvector there are also the five properties written down in the following last five expressions for this section:

$$\varepsilon\phi_h^{0(9)} + \alpha\psi_h^{0(9)} + \zeta\Phi_h^{0(9)} + \xi\Psi_h^{0(9)} = -\varepsilon K_E\phi_h^{0(7)} - \alpha K_A\psi_h^{0(7)} - \zeta K_Z\Phi_h^{0(7)} - \xi K_S\Psi_h^{0(7)} = \frac{e}{CK_{emgc}^2}\left(e\phi_h^{0(7)} + h\psi_h^{0(7)} + g\Phi_h^{0(7)} + f\Psi_h^{0(7)}\right) = \frac{eM_2}{CK_{emgc}^2} \quad (88)$$

$$\alpha K_A\phi_h^{0(9)} + \mu K_M\psi_h^{0(9)} + \beta K_B\Phi_h^{0(9)} + \lambda K_L\Psi_h^{0(9)} = 0 \quad (89)$$

Utilizing expression (89), the following equalities in expression (90) can be obtained:

$$\alpha\phi_h^{0(9)} + \mu\psi_h^{0(9)} + \beta\Phi_h^{0(9)} + \lambda\Psi_h^{0(9)} = \alpha\frac{K_\alpha^2}{K_{emgc}^2}\phi_h^{0(9)} + \mu\frac{K_m^2}{K_{emgc}^2}\psi_h^{0(9)} + \beta\frac{K_\beta^2}{K_{emgc}^2}\Phi_h^{0(9)} + \lambda\frac{K_\lambda^2}{K_{emgc}^2}\Psi_h^{0(9)} = \frac{h}{CK_{emgc}^2}\left(e\phi_h^{0(9)} + h\psi_h^{0(9)} + g\Phi_h^{0(9)} + f\Psi_h^{0(9)}\right) = \frac{h}{CK_{emgc}^2}\left(e\phi_h^{0(7)} + h\psi_h^{0(7)} + g\Phi_h^{0(7)} + f\Psi_h^{0(7)}\right) = \alpha\frac{K_\alpha^2}{K_{emgc}^2}\phi_h^{0(7)} + \mu\frac{K_m^2}{K_{emgc}^2}\psi_h^{0(7)} + \beta\frac{K_\beta^2}{K_{emgc}^2}\Phi_h^{0(7)} + \lambda\frac{K_\lambda^2}{K_{emgc}^2}\Psi_h^{0(7)} = \frac{hM_2}{CK_{emgc}^2} \quad (90)$$

For this section, the last two natural linear combinations demonstrate the following properties:

$$\zeta\phi_h^{0(9)} + \beta\psi_h^{0(9)} + \gamma\Phi_h^{0(9)} + \vartheta\Psi_h^{0(9)} = -\zeta K_Z\phi_h^{0(7)} - \beta K_B\psi_h^{0(7)} - \gamma K_G\Phi_h^{0(7)} - \vartheta K_T\Psi_h^{0(7)} = \frac{g}{CK_{emgc}^2}\left(e\phi_h^{0(7)} + h\psi_h^{0(7)} + g\Phi_h^{0(7)} + f\Psi_h^{0(7)}\right) = \frac{gM_2}{CK_{emgc}^2} \quad (91)$$

$$\xi\phi_h^{0(9)} + \lambda\psi_h^{0(9)} + \vartheta\Phi_h^{0(9)} + \eta\Psi_h^{0(9)} = -\xi K_S\phi_h^{0(7)} - \lambda K_L\psi_h^{0(7)} - \vartheta K_T\Phi_h^{0(7)} - \eta K_F\Psi_h^{0(7)} = \frac{f}{CK_{emgc}^2}\left(e\phi_h^{0(7)} + h\psi_h^{0(7)} + g\Phi_h^{0(7)} + f\Psi_h^{0(7)}\right) = \frac{fM_2}{CK_{emgc}^2} \quad (92)$$

6 The *E*-eigenvectors and their properties

The last independent pair of the eigenvectors is called the *E*-eigenvectors. Expressions (93) and (94) written below stand for the first and second *E*-eigenvectors, respectively. The corresponding components of the first *E*-eigenvector are naturally equal to parameters M_{ee} (22), M_{eh} (26), M_{eg} (27), and M_{ef} (28). Here, subscript "*e*" symbolizes the *E*-eigenvectors. Therefore, the *E*-eigenvectors are

$$\begin{pmatrix} U^{0(1)} \\ \phi^{0(1)} \\ \psi^{0(1)} \\ \Phi^{0(1)} \\ \Psi^{0(1)} \end{pmatrix} = \begin{pmatrix} U^{0(2)} \\ \phi^{0(2)} \\ \psi^{0(2)} \\ \Phi^{0(2)} \\ \Psi^{0(2)} \end{pmatrix} = \begin{pmatrix} U^{0(3)} \\ \phi^{0(3)} \\ \psi^{0(3)} \\ \Phi^{0(3)} \\ \Psi^{0(3)} \end{pmatrix} = \begin{pmatrix} U^{0(4)} \\ \phi^{0(4)} \\ \psi^{0(4)} \\ \Phi^{0(4)} \\ \Psi^{0(4)} \end{pmatrix} = \begin{pmatrix} U^{0(5)} \\ \phi^{0(5)} \\ \psi^{0(5)} \\ \Phi^{0(5)} \\ \Psi^{0(5)} \end{pmatrix} = \begin{pmatrix} U^{0(6)} \\ \phi^{0(6)} \\ \psi^{0(6)} \\ \Phi^{0(6)} \\ \Psi^{0(6)} \end{pmatrix} =$$

$$\begin{pmatrix} U^{0(7)} \\ \phi^{0(7)} \\ \psi^{0(7)} \\ \Phi^{0(7)} \\ \Psi^{0(7)} \end{pmatrix} = \begin{pmatrix} U^{0(8)} \\ \phi^{0(8)} \\ \psi^{0(8)} \\ \Phi^{0(8)} \\ \Psi^{0(8)} \end{pmatrix} = \begin{pmatrix} U_e^{0(7)} = 0 \\ \phi_e^{0(7)} = M_{ee} \\ \psi_e^{0(7)} = M_{eh} \\ \Phi_e^{0(7)} = M_{eg} \\ \Psi_e^{0(7)} = M_{ef} \end{pmatrix} \tag{93}$$

$$\begin{pmatrix} U^{0(9)} \\ \phi^{0(9)} \\ \psi^{0(9)} \\ \Phi^{0(9)} \\ \Psi^{0(9)} \end{pmatrix} = \begin{pmatrix} U^{0(10)} \\ \phi^{0(10)} \\ \psi^{0(10)} \\ \Phi^{0(10)} \\ \Psi^{0(10)} \end{pmatrix} = \begin{pmatrix} U_e^{0(9)} \\ \phi_e^{0(9)} \\ \psi_e^{0(9)} \\ \Phi_e^{0(9)} \\ \Psi_e^{0(9)} \end{pmatrix} \tag{94}$$

where

$$U_e^{0(9)} = \frac{e\phi_e^{0(9)} + h\psi_e^{0(9)} + g\Phi_e^{0(9)} + f\Psi_e^{0(9)}}{CK_{emgc}^2} = \frac{M_1}{CK_{emgc}^2} \tag{95}$$

$$\phi_e^{0(9)} = 2\beta\vartheta\lambda K_B K_T K_L + \mu\gamma\eta K_M K_G K_F - \gamma\lambda^2 K_G {K_L}^2 - \eta\beta^2 K_F {K_B}^2 - \mu\vartheta^2 K_M {K_T}^2 = \phi_e^{0(7)} - 2\beta\vartheta\lambda\left(\frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2}\right) - \mu\gamma\eta\left(\frac{K_m^2}{K_{emgc}^2} + \frac{K_g^2}{K_{emgc}^2} + \frac{K_f^2}{K_{emgc}^2}\right) + \gamma\lambda^2\left(\frac{K_g^2}{K_{emgc}^2} + 2\frac{K_\lambda^2}{K_{emgc}^2}\right) + \eta\beta^2\left(\frac{K_f^2}{K_{emgc}^2} + 2\frac{K_\beta^2}{K_{emgc}^2}\right) + \mu\vartheta^2\left(\frac{K_m^2}{K_{emgc}^2} + 2\frac{K_\vartheta^2}{K_{emgc}^2}\right) \quad (96)$$

$$\psi_e^{0(9)} = \alpha\vartheta^2 K_A {K_T}^2 + \eta\beta\zeta K_F K_B K_Z + \gamma\lambda\xi K_G K_L K_S - \gamma\eta\alpha K_G K_F K_A - \beta\vartheta\xi K_B K_T K_S - \vartheta\lambda\zeta K_T K_L K_Z = \psi_e^{0(7)} - \alpha\vartheta^2\left(\frac{K_\alpha^2}{K_{emgc}^2} + 2\frac{K_\vartheta^2}{K_{emgc}^2}\right) - \eta\beta\zeta\left(\frac{K_f^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) - \gamma\lambda\xi\left(\frac{K_g^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) + \gamma\eta\alpha\left(\frac{K_g^2}{K_{emgc}^2} + \frac{K_f^2}{K_{emgc}^2} + \frac{K_\alpha^2}{K_{emgc}^2}\right) + \beta\vartheta\xi\left(\frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) + \vartheta\lambda\zeta\left(\frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) \quad (97)$$

$$\Phi_e^{0(9)} = \eta\alpha\beta K_F K_A K_B + \lambda^2\zeta {K_L}^2 K_Z + \mu\vartheta\xi K_M K_T K_S - \beta\lambda\xi K_B K_L K_S - \alpha\vartheta\lambda K_A K_T K_L - \mu\eta\zeta K_M K_F K_Z = \Phi_e^{0(7)} - \eta\alpha\beta\left(\frac{K_f^2}{K_{emgc}^2} + \frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2}\right) - \lambda^2\zeta\left(2\frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) - \mu\vartheta\xi\left(\frac{K_m^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) + \beta\lambda\xi\left(\frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) + \alpha\vartheta\lambda\left(\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2}\right) + \mu\eta\zeta\left(\frac{K_m^2}{K_{emgc}^2} + \frac{K_f^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) \quad (98)$$

$$\Psi_e^{0(9)} = \gamma\alpha\lambda K_G K_A K_L + \mu\vartheta\zeta K_M K_T K_Z + \beta^2\xi {K_B}^2 K_S - \alpha\beta\vartheta K_A K_B K_T - \beta\lambda\zeta K_B K_L K_Z - \mu\gamma\xi K_M K_G K_S = \Psi_e^{0(7)} - \gamma\alpha\lambda\left(\frac{K_g^2}{K_{emgc}^2} + \frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2}\right) - \mu\vartheta\zeta\left(\frac{K_m^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) - \beta^2\xi\left(2\frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) + \alpha\beta\vartheta\left(\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2}\right) + \beta\lambda\zeta\left(\frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) + \mu\gamma\xi\left(\frac{K_m^2}{K_{emgc}^2} + \frac{K_g^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) \quad (99)$$

With the natural linear combinations of the suitable *E*-eigenvectors' components, this pair of the *E*-eigenvectors possesses the properties demonstrated in the rest ten expressions written below, where M_1 in expression

(100) stands for CEMGCMC coupling mechanism (17). The following first five expressions relate to the linear combinations of the components of the first E-eigenvector:

$$e\phi_e^{0(7)} + h\psi_e^{0(7)} + g\Phi_e^{0(7)} + f\Psi_e^{0(7)} = e\phi_e^{0(9)} + h\psi_e^{0(9)} + g\Phi_e^{0(9)} + f\Psi_e^{0(9)} = eM_{ee} + hM_{eh} + gM_{eg} + fM_{ef} = M_1 \quad (100)$$

$$\varepsilon\phi_e^{0(7)} + \alpha\psi_e^{0(7)} + \zeta\Phi_e^{0(7)} + \xi\Psi_e^{0(7)} = \varepsilon M_{ee} + \alpha M_{eh} + \zeta M_{eg} + \xi M_{ef} = M_5 \quad (101)$$

$$\alpha\phi_e^{0(7)} + \mu\psi_e^{0(7)} + \beta\Phi_e^{0(7)} + \lambda\Psi_e^{0(7)} = \alpha K_A\phi_e^{0(9)} + \mu K_M\psi_e^{0(9)} + \beta K_B\Phi_e^{0(9)} + \lambda K_L\Psi_e^{0(9)} = 0 \quad (102)$$

$$\zeta\phi_e^{0(7)} + \beta\psi_e^{0(7)} + \gamma\Phi_e^{0(7)} + \vartheta\Psi_e^{0(7)} = \zeta K_Z\phi_e^{0(9)} + \beta K_B\psi_e^{0(9)} + \gamma K_G\Phi_e^{0(9)} + \vartheta K_T\Psi_e^{0(9)} = 0 \quad (103)$$

$$\xi\phi_e^{0(7)} + \lambda\psi_e^{0(7)} + \vartheta\Phi_e^{0(7)} + \eta\Psi_e^{0(7)} = \xi K_S\phi_e^{0(9)} + \lambda K_L\psi_e^{0(9)} + \vartheta K_T\Phi_e^{0(9)} + \eta K_F\Psi_e^{0(9)} = 0 \quad (104)$$

Concerning the properties of the second E-eigenvector, the following equality is useful:

$$\varepsilon K_E\phi_e^{0(9)} + \alpha K_A\psi_e^{0(9)} + \zeta K_Z\Phi_e^{0(9)} + \xi K_S\Psi_e^{0(9)} = 0 \quad (105)$$

With expressions (105), (32), (36), (39), and (40), the following property can be obtained:

$$\varepsilon\phi_e^{0(9)} + \alpha\psi_e^{0(9)} + \zeta\Phi_e^{0(9)} + \xi\Psi_e^{0(9)} = \varepsilon\frac{K_E^2}{K_{emgc}^2}\phi_e^{0(9)} + \alpha\frac{K_A^2}{K_{emgc}^2}\psi_e^{0(9)} + \zeta\frac{K_\zeta^2}{K_{emgc}^2}\Phi_e^{0(9)} + \xi\frac{K_\xi^2}{K_{emgc}^2}\Psi_e^{0(9)} = \frac{e}{CK_{emgc}^2}\left(e\phi_e^{0(9)} + h\psi_e^{0(9)} + g\Phi_e^{0(9)} + \right.$$

$$f\Psi_e^{0(9)}\Big) = \frac{e}{CK_{emgc}^2}\Big(e\phi_e^{0(7)} + h\psi_e^{0(7)} + g\Phi_e^{0(7)} + f\Psi_e^{0(7)}\Big) = \varepsilon\frac{K_e^2}{K_{emgc}^2}\phi_e^{0(7)} + \alpha\frac{K_\alpha^2}{K_{emgc}^2}\psi_e^{0(7)} + \zeta\frac{K_\zeta^2}{K_{emgc}^2}\Phi_e^{0(7)} + \xi\frac{K_\xi^2}{K_{emgc}^2}\Psi_e^{0(7)} = \frac{eM_1}{CK_{emgc}^2} \quad (106)$$

The final three expressions for the linear combinations of the components of the second E-eigenvector are

$$\alpha\phi_e^{0(9)} + \mu\psi_e^{0(9)} + \beta\Phi_e^{0(9)} + \lambda\Psi_e^{0(9)} = -\alpha K_A\phi_e^{0(7)} - \mu K_M\psi_e^{0(7)} - \beta K_B\Phi_e^{0(7)} - \lambda K_L\Psi_e^{0(7)} = \frac{h}{CK_{emgc}^2}\Big(e\phi_e^{0(7)} + h\psi_e^{0(7)} + g\Phi_e^{0(7)} + f\Psi_e^{0(7)}\Big) = \frac{hM_1}{CK_{emgc}^2} \quad (107)$$

$$\zeta\phi_e^{0(9)} + \beta\psi_e^{0(9)} + \gamma\Phi_e^{0(9)} + \vartheta\Psi_e^{0(9)} = -\zeta K_Z\phi_e^{0(7)} - \beta K_B\psi_e^{0(7)} - \gamma K_G\Phi_e^{0(7)} - \vartheta K_T\Psi_e^{0(7)} = \frac{g}{CK_{emgc}^2}\Big(e\phi_e^{0(7)} + h\psi_e^{0(7)} + g\Phi_e^{0(7)} + f\Psi_e^{0(7)}\Big) = \frac{gM_1}{CK_{emgc}^2} \quad (108)$$

$$\xi\phi_e^{0(9)} + \lambda\psi_e^{0(9)} + \vartheta\Phi_e^{0(9)} + \eta\Psi_e^{0(9)} = -\xi K_S\phi_e^{0(7)} - \lambda K_L\psi_e^{0(7)} - \vartheta K_T\Phi_e^{0(7)} - \eta K_F\Psi_e^{0(7)} = \frac{f}{CK_{emgc}^2}\Big(e\phi_e^{0(7)} + h\psi_e^{0(7)} + g\Phi_e^{0(7)} + f\Psi_e^{0(7)}\Big) = \frac{fM_1}{CK_{emgc}^2} \quad (109)$$

So, this paper has obtained the four pairs of the main independent eigenvectors called the F-eigenvectors (third section), G-eigenvectors (fourth section), H-eigenvectors (fifth section), and E-eigenvectors (this sixth section). Their properties are described in the corresponding section below each pair of the corresponding eigenvectors. All the four pairs of the main independent eigenvectors are written above in convenient explicit forms. However, the reader can find that their forms are quite complicated and to verify their properties is not easy to perform. Indeed, it was not easy to find that all the twenty-four possible eigenvectors (paper [24] has obtained only half of them) reduce to one of the four pairs of the main independent eigenvectors and then to demonstrate the properties of the F-, G-, H-, and E-eigenvectors.

All the properties obtained in this and the previous three sections are the main results of these theoretical investigations. In this section, property (100) demonstrates that the E-eigenvectors relate to the coupling mechanism M_1 (17) of the CEMGCMC K_{emgc}^2 (16). Corresponding properties (49), (66), and (83) in

the previous three sections manifest that the *F*-, *G*-, and *H*-eigenvectors relate to the CEMGCMC coupling mechanisms M_4, M_3, and M_2 defined by formulae (20), (19), and (18), respectively. Moreover, all of these four independent eigenvectors also relate to the fifth coupling mechanisms M_5 (21). Properties (53), (69), (85), and (101) support this statement. It is necessary to state that each of these five coupling mechanisms can release propagation velocities of the 4P-SH acoustic waves guided by the surface of the solid, the interface between two dissimilar solids, or plate (thin film) waveguide. All the properties of the four eigenvectors obtained in this and the previous three sections can be actually useful for analysis of various boundary conditions' determinants to obtain suitable propagation velocities. Indeed, suitable properties can be used for representing final results of propagation speeds in compact and convenient forms because many linear combinations of the eigenvector components (i.e. properties) are equal to zero. This will be useful even in numerical calculations because it is obvious that many possible final results of propagation speeds will be very complicated due to the existence of a lot of material parameters listed in table 1. Also, for the calculations it is preferable to use measured values of the parameters listed in the table. Note that measured values of the material parameters listed in the last several rows of table 1 are currently absent because their measurements require creation of perfect experimental tools.

It is necessary to mention that the acoustic waves are very slow in comparison with the speed of light, C_L, and can propagate in solids at the speeds of $\sim 10^3$ m/s. For the mechanical subsystem, the treatment of additional subsystems such as the electrical, magnetic, gravitational, and cogravitational can result in the fact that the propagation speeds of the acoustic waves can be slightly higher. However, the presence of these additional subsystems allows one to deal with smart materials because any change in one subsystem can result in some response changes in the other subsystems. This fact allows the constitution of various smart technical devices.

7 Discussion

It is necessary to mention also that this work belongs to the case of investigations of the piezogravitocogravitoelectromagnetic materials [1, 31] possessing the mechanical, electrical, magnetic, gravitational, and cogravitational subsystems. Namely this paper further develops the results obtained in pioneer work [1] and paper [24]. It is now possible to compare the

obtained results in this work with the results obtained in book [12] for the case of acoustic wave propagation in piezoelectromagnetics (PEMs) possessing only the mechanical, electrical, and magnetic subsystems, i.e. without taking into account the gravitational phenomena. For the case of the PEM materials there are only two possible pairs of the main independent eigenvectors and they are quite simple compared with the case described in this paper above. However, the investigations of the PEM materials also have peculiarities described in papers [32] and [33]. To ignore these peculiarities can result in obtaining incorrect final solution for the acoustic wave propagation velocities in the PEM materials. The reader can also find single review [14] on the subject of the acoustic wave propagation in the PEM materials. The common problem representing the peculiarity of this study and study [12] is that any eigenvector component does not depend on the propagation velocity. It is worth noting that the cubic PEM materials do not possess such peculiarity because the eigenvectors depend on the propagation velocity in the cubic case treated in book [13]. However, the cubic PEM materials have the other peculiarities. Concerning some investigations of the piezogravitocogravitoelectromagnetic materials with the cubic symmetry, the development of theoretical backgrounds is also possible in the future.

It is now possible to briefly discuss the most exciting thing for the reader concerning the possibility of the instant interplanetary (tele)communication. It was mentioned in the introduction that in the solids, the slow speed of the new 4P-SH-SAW [1] can naturally depend on the speed of the electromagnetic wave, speed of the gravitational wave, and the speeds $\Lambda_{01} = (\zeta_0\lambda_0)^{-1/2} \rightarrow 10^{13}C_L$ and $\Lambda_{02} = (\xi_0\beta_0)^{-1/2} \rightarrow 10^{13}C_L$ of the new fast waves that can be thirteen orders faster than the speed of light, C_L. Also, it is necessary to state that any acoustic wave cannot propagate in a vacuum. So, the transition of information in a vacuum is only possible with one of the following speeds: C_L, Λ_{01}, and Λ_{02}. It is obvious that the speed C_L is too slow for the instant interplanetary communication. On the other hand, the speeds Λ_{01}, and Λ_{02} that thirteen orders faster than C_L are already enough for the instant interplanetary (and even intergalactic) communication. Therefore, it is necessary to develop technologies and infrastructure in order to have the instant interplanetary communication. Indeed, it is not easy and fast. First of all, it is necessary to measure the vacuum parameters ζ_0, λ_0, ξ_0, and β_0 called the gravitoelectric, cogravitomagnetic, cogravitoelectric, and gravitomagnetic constants, respectively. This will allow the human civilization

to know the complete set of the vacuum parameters and the exact values of the speeds A_{01} and A_{02}.

The C_L-communication due to the well-developed C_L-infrastructure is well-developed on the Earth. This successful development lasted during the last several centuries and involved millions of researchers and engineers on this planet. This means that trillions of the United States dollars were accumulated to have the success in the development of the C_L-communication that is now used even for the slow interplanetary communication. The C_L-communication simultaneously incorporates only two subsystems (electrical and magnetic) to transfer signals through a vacuum. For the mobile C_L-communication, many kinds of mobile devices are commercially available today, for instance, mobile phones even for children. For the A_{01}- and A_{02}-communication (say A-communication below) it is possible to use some interplanetary infrastructure schematically shown and discussed in paper [10]. It is assumed that the development of some mobile A-communication can be the final aim after the successful development of the voice and video A-communications. For this purpose, it is useful to know the complete set of the material constants for solids listed in table 1.

There are millions of different solids (monocrystals, alloys, composites, etc.) and their wave characteristics can be very important to constitute various (smart) technical devices. However, it is necessary to state that the contributions from both the piezoelectric and piezomagnetic effects to the speed of the acoustic wave are in general small but well-measurable today. Even the contribution from the magnetoelectric effect for the smart PEM materials is well-measurable today in spite of its weakness. The magnetoelectric effect is characterized by the electromagnetic constant α listed in table 1. This constant symbolizes some exchange between the electrical and magnetic subsystems with the following evaluated exchange speed: $V_\alpha = 1/\alpha$. The exchange speed V_α must be faster than the speed of the electromagnetic wave V_{EM} in a PEM solid, $V_\alpha > V_{EM} = (\varepsilon\mu)^{-1/2}$ because the following condition of thermodynamic stability $\alpha^2 < \varepsilon\mu$ [34, 35] must hold. Practically there is always $\alpha^2 << \varepsilon\mu$ ($V_\alpha >> V_{EM}$) for the magnetoelectric solids. Even for strong magnetoelectric composites the speed V_α is several orders faster than the speed V_{EM}. For majority of the magnetoelectric solids, it is assumed that the difference between the speeds V_α and V_{EM} can even reach ten orders. Indeed, this is a huge difference. The thermodynamic stability condition $V_\alpha >> V_{EM}$ for the system representing the electromagnetic wave in a solid or a

vacuum means that the exchange speed V_α between the electrical and magnetic subsystems should be significantly higher the speed V_{EM} of the whole system. This allows the whole system to thermodynamically exist and propagate for long distances in continuous media such as a vacuum and solids.

For the gravitational and cogravitational subsystems there is the gravitocogravitic constant ϑ listed in table 1 and the evaluated exchange speed is $V_\vartheta = 1/\vartheta$. The corresponding thermodynamic stability condition can be written as follows: $\vartheta^2 < \gamma\eta$ and $V_\vartheta > V_{GC} = (\gamma\eta)^{-1/2}$ (even $\vartheta^2 << \gamma\eta$ and $V_\vartheta >> V_{GC}$) where the gravitic constant γ and the cogravitic constant η are also listed in table 1. It is natural to expect that minimum values for the exchange speeds V_α and V_ϑ can be comparable. Some exchange possibilities must exist among the gravitational, cogravitational, electrical, and magnetic subsystems. The possible exchange processes between each pair of the four subsystems can be characterized by the material constants ζ, λ, ξ, and β listed in the last four rows of table 1. These constants introduce the following exchange speeds: $\Lambda_1 = (\zeta\lambda)^{-1/2} \rightarrow 10^{13}C_L$ and $\Lambda_2 = (\xi\beta)^{-1/2} \rightarrow 10^{13}C_L$. The thermodynamic stability conditions can be written here as follows: $\Lambda_1 > V_\alpha > V_{EM}$, $\Lambda_2 > V_\alpha > V_{EM}$, $\Lambda_1 > V_\vartheta > V_{GC}$, $\Lambda_2 > V_\vartheta > V_{GC}$ or even $\Lambda_1 >> V_\alpha >> V_{EM}$, $\Lambda_2 >> V_\alpha >> V_{EM}$, $\Lambda_1 >> V_\vartheta >> V_{GC}$, $\Lambda_2 >> V_\vartheta >> V_{GC}$. The contributions from all of these speeds to the speed of the acoustic wave defined by formula (12) are complicatedly hybridized in the CEMGCMC K^2_{emgc} (13) that possesses the five different coupling mechanisms in form (16).

For a vacuum, it is also possible to assume that there are the following thermodynamic stability conditions: $\Lambda_{01} > V_{\alpha 0} > C_L$, $\Lambda_{02} > V_{\alpha 0} > C_L$, $\Lambda_{01} > V_{\vartheta 0} > C_L$, $\Lambda_{02} > V_{\vartheta 0} > C_L$ or even $\Lambda_{01} >> V_{\alpha 0} >> C_L$, $\Lambda_{02} >> V_{\alpha 0} >> C_L$, $\Lambda_{01} >> V_{\vartheta 0} >> C_L$, $\Lambda_{02} >> V_{\vartheta 0} >> C_L$, where $V_{\alpha 0} = 1/\alpha_0$ and $V_{\vartheta 0} = 1/\vartheta_0$. The most popular case of interactions among the gravitational, cogravitational, electrical, and magnetic subsystems in a vacuum is the captivation of an electromagnetic wave propagating close enough to a gigantic black hole. This case was also discussed in paper [31]. In this case, the black hole gravitation can change the propagation direction of the electromagnetic wave and the wave will be finally directed towards the black hole. It is obvious that these interactions must be significantly faster than the speed of light, C_L, i.e. at the fast speeds Λ_{01} and Λ_{02}. This allows one to treat the system (the electromagnetic wave) as a stationary system at an equilibrium. Each interaction is negligibly small that allows the system to relax after each exchange among the gravitational, cogravitational, electrical, and magnetic fields. However, an enormous number of these interactions can result

in significant change in the propagation direction of the electromagnetic wave while the propagation speed C_L can stay the same because a vacuum represents an isotropic continuum, i.e. the value of the speed of light is the same in any propagation direction. Also, it is assumed that some change in energy of the electromagnetic wave can exist.

8 Conclusion

This theoretical research has obtained the *E*-, *H*-, *G*-, and *F*-eigenvectors. They are the four pairs of the main independent eigenvectors. The properties of the obtained eigenvectors were also demonstrated. It was found that some suitable linear combinations of the eigenvector components can be equal to one of the CEMGCMC coupling mechanisms M_1, M_2, M_3, M_4, and M_5 defined by expressions (17), (18), (19), (20), and (21), respectively. These eigenvectors (namely the CEMGCMC coupling mechanisms) can define the propagation velocities in the theory of the acoustic wave propagation in the transversely isotropic materials of the symmetry class 6 *mm*. The acoustic waves propagate in the directions when there is some coupling among the mechanical, electrical, magnetic, gravitational, and cogravitational subsystems. It is expected that the coupling with some gravitational phenomena will allow the human civilization to develop gravitoelectromagnetic technologies for instant interplanetary (interstellar, intergalactic) communications. Indeed, the instant intergalactic Internet is necessary that can be based on the new fast four-potential gravitocogravitoelectromagnetic waves instead of the conventional two-potential electromagnetic waves propagating at the speed of light. The cooperative development of the electromagnetic and gravitational technologies can be feasible already to the end of this century because the four-potential waves can propagate in a vacuum at the speed thirteen orders faster than the speed of light.

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ON n-MULTIPLICATIVE HIGHER DERIVATIONS ON ALTERNATIVE RINGS

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Abstract

Let $\mathfrak{R}$ be an alternative ring. We study the additivity of n-multiplicative higher derivations of $\mathfrak{R}$. We prove that if $\mathfrak{R}$ contains a family of nontrivial idempotents satisfying the Martindale's conditions, then these derivations are additive.

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1 Alternative rings and n-multiplicative higher derivations

Let $\mathfrak{R}$ be a ring not necessarily associative or commutative and consider the following convention for its multiplication operation: $xy \cdot z = (xy)z$ and $x \cdot yz = x(yz)$ for $x, y, z \in \mathfrak{R}$, to reduce the number of parentheses. We denote the *associator* of $\mathfrak{R}$ by $(x, y, z) = xy \cdot z - x \cdot yz$ for $x, y, z \in \mathfrak{R}$.

Let $X = \{x_i\}_{i \in \mathbb{N}}$ be an arbitrary set of variables. A *nonassociative monomial of degree* 1 is any element of X. Given a natural number $n > 1$, a *nonassociative monomial of degree* n is an expression of the form $(u)(v)$, where u is a nonassociative monomial of some degree i and v a nonassociative monomial of degree $n - i$. A *nonassociative polynomial* p over a ring $\mathfrak{R}$ is any formal linear combination of nonassociative monomials with coefficients in $\mathfrak{R}$. If p includes no variables except $x_1, x_2, \ldots, x_n$ and $a_1, a_2, \ldots, a_n$ is a set of elements of $\mathfrak{R}$, then $p(a_1, a_2, \ldots, a_n)$ is an element of $\mathfrak{R}$ which results by applying the sequence of operations forming p to $a_1, a_2, \ldots, a_n$ in place of $x_1, x_2, \ldots, x_n$.

Let $\mathfrak{R}$ be a ring and $d : \mathfrak{R} \to \mathfrak{R}$ a map of $\mathfrak{R}$ into $\mathfrak{R}$. We call d a *n-multiplicative derivation* of $\mathfrak{R}$ if

$$d\big(m(x_1, \cdots, x_n)\big) = \sum_{i=1}^{n} m\big(x_1, \cdots, d(x_i), \cdots, x_n\big)$$

for all nonassociative monomial $m = m(x_1, \cdots, x_n)$ of degree n and arbitrary elements $x_1, \cdots, x_n \in \mathfrak{R}$. Let $D = \{d_p\}_{p \in \mathbb{N}}$ be a family of maps $d_p : \mathfrak{R} \to \mathfrak{R}$, for all $p \in \mathbb{N}$, such that $d_0 = I_{\mathfrak{R}}$ (the identity map on $\mathfrak{R}$). We call D a *n-multiplicative higher derivation* if

$$d_p(m(x_1, \cdots, x_n)) = \sum_{p_1 + \cdots + p_n = p} m(d_{p_1}(x_1), \cdots, d_{p_n}(x_n)),$$

for all nonassociative monomial $m = m(x_1, \cdots, x_n)$ of degree n and arbitrary elements $x_1, \cdots, x_n \in \mathfrak{R}$, for each $p \in \mathbb{N}$. We say that a n-multiplicative higher derivation $D = \{d_p\}_{m \in \mathbb{N}}$ is *additive* if d_p is additive, for each $p \in \mathbb{N}$.

A ring $\mathfrak{R}$ is said to be *alternative* if $(x, x, y) = 0 = (y, x, x)$ for all $x, y \in \mathfrak{R}$. One easily sees that any associative ring is an alternative ring.

An alternative ring $\mathfrak{R}$ is called *k-torsion free* if $kx = 0$ implies $x = 0$, for any $x \in \mathfrak{R}$, where $k \in \mathbb{Z}$, $k > 0$, and *prime* if $\mathfrak{AB} \neq 0$ for any two nonzero ideals $\mathfrak{A}, \mathfrak{B} \subseteq \mathfrak{R}$.

A nonzero element $e_1 \in \mathfrak{R}$ is called an *idempotent* if $e_1e_1 = e_1$ and a *nontrivial idempotent* if it is an idempotent different from the multiplicative identity element of $\mathfrak{R}$. Let us consider $\mathfrak{R}$ an alternative ring and fix a nontrivial idempotent $e_1 \in \mathfrak{R}$. Let $e_2 \colon \mathfrak{R} \to \mathfrak{R}$ and $e_2' \colon \mathfrak{R} \to \mathfrak{R}$ be linear operators given by $e_2(a) = a - e_1a$ and $e_2'(a) = a - ae_1$. Clearly $e_2^2 = e_2$, $(e_2')^2 = e_2'$ and we note that if $\mathfrak{R}$ has a unity, then we can consider $e_2 = 1 - e_1 \in \mathfrak{R}$. Let us denote $e_2(a)$ by e_2a and $e_2'(a)$ by ae_2. It is easy to see that $e_ia \cdot e_j = e_i \cdot ae_j$ $(i, j = 1, 2)$ for all $a \in \mathfrak{R}$. Then $\mathfrak{R}$ has a Peirce decomposition $\mathfrak{R} = \mathfrak{R}_{11} \oplus \mathfrak{R}_{12} \oplus \mathfrak{R}_{21} \oplus \mathfrak{R}_{22}$, where $\mathfrak{R}_{ij} = e_i\mathfrak{R}e_j$ $(i, j = 1, 2)$ [1], satisfying the following multiplicative relations:

(i) $\mathfrak{R}_{ij}\mathfrak{R}_{jl} \subseteq \mathfrak{R}_{il}$ $(i, j, l = 1, 2)$;

(ii) $\mathfrak{R}_{ij}\mathfrak{R}_{ij} \subseteq \mathfrak{R}_{ji}$ $(i, j = 1, 2)$;

(iii) $\mathfrak{R}_{ij}\mathfrak{R}_{kl} = 0$, if $j \neq k$ and $(i, j) \neq (k, l)$, $(i, j, k, l = 1, 2)$;

(iv) $x_{ij}^2 = 0$, for all $x_{ij} \in \mathfrak{R}_{ij}$ $(i, j = 1, 2;\ i \neq j)$.

The first result about the additivity of multiplicative derivations on alternative rings was given by Rodrigues et al. [4]. The authors established conditions on a alternative ring $\mathfrak{R}$ such that every multiplicative derivation on $\mathfrak{R}$ is additive. Ferreira et al. [2] also considered this question in the context of n-multiplicative maps on alternative rings satisfying the Martindale's conditions [3]. In the context of n-multiplicative derivations they proved the following theorem.

Theorem 1.1. *[2, Theorem 2.3] Let $\mathfrak{R}$ be an alternative ring containing a family $\{e_\alpha | \alpha \in \Lambda\}$ of idempotents which satisfies:*

(i) If $x \in \mathfrak{R}$ is such that $x\mathfrak{R} = 0$, then $x = 0$;

(ii) If $x \in \mathfrak{R}$ is such that $e_\alpha\mathfrak{R} \cdot x = 0$ (or $e_\alpha \cdot \mathfrak{R}x = 0$) for all $\alpha \in \Lambda$, then $x = 0$ (and hence $\mathfrak{R}x = 0$ implies $x = 0$);

(iii) For each $\alpha \in \Lambda$ and $x \in \mathfrak{R}$, if $(e_\alpha x e_\alpha) \cdot \mathfrak{R}(1 - e_\alpha) = 0$ then $e_\alpha x e_\alpha = 0$.

Then every n-multiplicative derivation d of $\mathfrak{R}$ is additive.

In the present paper we generalize the Theorem 1.1. We investigate the problem of when a n-multiplicative higher derivation is additive for the class of alternative rings satisfying the Martindale's conditions.

2 Main theorem

We shall prove as follows the main result of this paper.

Theorem 2.1. *Let $\mathfrak{R}$ be an alternative ring containing a family $\{e_\alpha | \alpha \in \Lambda\}$ of idempotents which satisfies:*

(i) If $x \in \mathfrak{R}$ is such that $x\mathfrak{R} = 0$, then $x = 0$;

(ii) If $x \in \mathfrak{R}$ is such that $e_\alpha \mathfrak{R} \cdot x = 0$ (or $e_\alpha \cdot \mathfrak{R}x = 0$) for all $\alpha \in \Lambda$, then $x = 0$ (and hence $\mathfrak{R}x = 0$ implies $x = 0$);

(iii) For each $\alpha \in \Lambda$ and $x \in \mathfrak{R}$, if $(e_\alpha x e_\alpha) \cdot \mathfrak{R}(1 - e_\alpha) = 0$ then $e_\alpha x e_\alpha = 0$.

Then every n-multiplicative higher derivation $D = \{d_p\}_{p \in \mathbb{N}}$ of $\mathfrak{R}$ is additive.

Note that the map d_1 is additive, by Theorem 1.1. We will use this result throughout the paper whenever needed without specific mention.

Before beginning with the demonstration of theorem we will present a Lemma which will aid us in its demonstration.

Lemma 2.1. *$d_p(0) = 0$, for each $p \in \mathbb{N}$.*

Proof. By induction hypothesis, suppose that $d_k(0) = 0$ for all positive integer $k < p$. Then

$$\begin{aligned} & d_p(0) \\ = \; & d_p(0(\cdots(00)\cdots)) \\ = \; & \sum_{p_1+\cdots+p_n=p} d_{p_1}(0)(\cdots(d_{p_{n-1}}(0)d_{p_n}(0))\cdots) \\ = \; & 0(\cdots(0d_p(0))\cdots) \end{aligned}$$

$$+ \sum_{\substack{p_1+\cdots+p_n=p \\ p_n\neq p}} d_{p_1}(0)(\cdots(d_{p_{n-1}}(0)d_{p_n}(0))\cdots)$$
$$= 0.$$

Thus, the result is true for the positive integer p and holds the induction hypothesis. Therefore, by the mathematical induction principle, we can conclude that $d_p(0) = 0$ is true for each $p \in \mathbb{N}$. □

Let $\mathfrak{R}$ an alternative ring and $d : \mathfrak{R} \to \mathfrak{R}$ a map. Let us set $f(x,y) = d(x+y) - d(x) - d(y)$ for all $x, y \in \mathfrak{R}$. Then, we see that $f(x,y) = 0$ if and only if d is additive. This technique allows the discussion of the additivity of n-multiplicative higher derivations on alternative rings, from the result [2, Theorem 2.1].

Proof of the Theorem 2.1. By induction hypothesis, suppose that d_k is additive for each positive integer $k < p$. Then for arbitrary elements $x, y \in \mathfrak{R}$, we set $f(x,y) = d_p(x+y) - d_p(x) - d_p(y)$. It easily follows that $f(x,0) = 0 = f(0,x)$ for all $x \in \mathfrak{R}$, by Lemma 2.1. We will show that $f(x,y)u = f(xu,yu)$ and $uf(x,y) = f(ux,uy)$ for all $u, x, y \in \mathfrak{R}$. We consider only the cases $n = 3$ and $n > 3$, since the case $n = 2$ is is easily verifiable. Thus, for arbitrary elements $r, u \in \mathfrak{R}$, we have:
Case $(n = 3)$. First, let us observe that by direct calculation we have

$$\begin{aligned}
& r \cdot f(x,y)u \\
= \; & r \cdot \big(d_p(x+y) - d_p(x) - d_p(y)\big)u \\
= \; & r \cdot d_p(x+y)u - r \cdot d_p(x)u - r \cdot d_p(y)u \\
= \; & d_p\big(r \cdot (x+y)u\big) - \sum_{\substack{p_1+p_2+p_3=p \\ p_2\neq p}} d_{p_1}(r) \cdot d_{p_2}(x+y)d_{p_3}(u) \\
& -d_p(r \cdot xu) + \sum_{\substack{p_1+p_2+p_3=p \\ p_2\neq p}} d_{p_1}(r) \cdot d_{p_2}(x)d_{p_3}(u) \\
& -d_p(r \cdot yu) + \sum_{\substack{p_1+p_2+p_3=p \\ p_2\neq p}} d_{p_1}(r) \cdot d_{p_2}(y)d_{p_3}(u) \\
= \; & d_p\big(r(xu+yu)\big) - d_p(r \cdot xu) - d_p(r \cdot yu).
\end{aligned}$$

It follows that

$$
\begin{aligned}
& e_\alpha \cdot r\big(f(x,y)u\big) \\
= \;& e_\alpha \cdot e_\alpha\big(r\big(f(x,y)u\big)\big) \\
= \;& e_\alpha \cdot e_\alpha\big(d_p\big(r(xu+yu)\big) - d_p\big(r\cdot xu\big) - d_p\big(r\cdot yu\big)\big) \\
= \;& e_\alpha \cdot e_\alpha d_p\big(r(xu+yu)\big) - e_\alpha \cdot e_\alpha d_p\big(r\cdot xu\big) - e_\alpha \cdot e_\alpha d_p\big(r\cdot yu\big) \\
= \;& d_p\big(e_\alpha \cdot e_\alpha\big(r(xu+yu)\big)\big) \\
& \qquad - \sum_{\substack{p_1+p_2+p_3=p \\ p_3\neq p}} d_{p_1}(e_\alpha)\cdot d_{p_2}(e_\alpha) d_{p_3}\big(r(xu+yu)\big) \\
& -d_p\big(e_\alpha \cdot e_\alpha(r\cdot xu)\big) + \sum_{\substack{p_1+p_2+p_3=p \\ p_3\neq p}} d_{p_1}(e_\alpha)\cdot d_{p_2}(e_\alpha) d_{p_3}(r\cdot xu) \\
& -d_p\big(e_\alpha \cdot e_\alpha(r\cdot yu)\big) + \sum_{\substack{p_1+p_2+p_3=p \\ p_3\neq p}} d_{p_1}(e_\alpha)\cdot d_{p_2}(e_\alpha) d_{p_3}(r\cdot yu) \\
= \;& d_p\big(e_\alpha \cdot e_\alpha\big(r(xu+yu)\big)\big) - d_p\big(e_\alpha \cdot e_\alpha(r\cdot xu)\big) - d_p\big(e_\alpha \cdot e_\alpha(r\cdot yu)\big) \\
= \;& d_p\big(e_\alpha \cdot \big(r(xu+yu)\big)\big) - d_p\big(e_\alpha \cdot (r\cdot xu)\big) - d_p\big(e_\alpha (r\cdot yu)\big) \\
= \;& \sum_{p_1+p_2+p_3=p} d_{p_1}(e_\alpha)\cdot d_{p_2}(r) d_{p_3}(xu+yu) \\
& - \sum_{p_1+p_2+p_3=p} d_{p_1}(e_\alpha)\cdot d_{p_2}(r) d_{p_3}(xu) \\
& - \sum_{p_1+p_2+p_3=p} d_{p_1}(e_\alpha)\cdot d_{p_2}(r) d_{p_3}(yu) \\
= \;& e_\alpha \cdot r d_p(xu+yu) - e_\alpha \cdot r d_p(xu) - e_\alpha \cdot r d_p(yu) \\
= \;& e_\alpha \cdot r\big(d_p(xu+yu) - d_p(xu) - d_p(yu)\big) \\
= \;& e_\alpha \cdot r f(xu,yu)
\end{aligned}
$$

for all $\alpha \in \Lambda$. Hence, $e_\alpha \cdot r\big(f(x,y)u - f(xu,yu)\big) = 0$ for all $\alpha \in \Lambda$ and arbitrary elements $r, x, y \in \mathfrak{R}$. This implies $e_\alpha \cdot \mathfrak{R}\big(f(x,y)u - f(xu,yu)\big) = 0$ which results in $f(x,y)u = f(xu,yu)$, by condition *(ii)* of the Theorem. Similarly, we prove $uf(x,y) = f(ux,uy)$.

Case $(n > 3)$.

$$
\begin{aligned}
& e_\alpha \cdot r\big(f(x,y)u\big) \\
= \; & \underbrace{e_\alpha\Big(\cdots e_\alpha}_{(n-3)\text{ times}}\Big(r\big(f(x,y)u\big)\Big)\cdots\Big) \\
= \; & \underbrace{e_\alpha\Big(\cdots e_\alpha}_{(n-3)\text{ times}}\Big(r\Big(\big(d_p(x+y)-d_p(x)-d_p(y)\big)u\Big)\Big)\cdots\Big) \\
= \; & \underbrace{e_\alpha\Big(\cdots e_\alpha}_{(n-3)\text{ times}}\Big(r\big(d_p(x+y)u\big)\Big)\cdots\Big) - \underbrace{e_\alpha\Big(\cdots e_\alpha}_{(n-3)\text{ times}}\Big(r\big(d_p(x)u\big)\Big)\cdots\Big) \\
& - \underbrace{e_\alpha\Big(\cdots e_\alpha}_{(n-3)\text{ times}}\Big(r\big(d_p(y)u\big)\Big)\cdots\Big) \\
= \; & d_p\Big(\underbrace{e_\alpha\Big(\cdots e_\alpha}_{(n-3)\text{ times}}\Big(r\big((x+y)u\big)\Big)\cdots\Big)\Big) \\
& - \sum_{\substack{p_1+\cdots+p_n=p \\ p_{n-1}\neq p}} \underbrace{d_{p_1}(e_\alpha)\Big(\cdots d_{p_{n-3}}(e_\alpha)}_{(n-3)\text{ times}} \\
& \qquad\qquad \Big(d_{p_{n-2}}(r)\big(d_{p_{n-1}}(x+y)d_{p_n}(u)\big)\Big)\cdots\Big) \\
& - d_p\Big(\underbrace{e_\alpha\Big(\cdots e_\alpha}_{(n-3)\text{ times}}\big(r(xu)\big)\cdots\Big)\Big) \\
& + \sum_{\substack{p_1+\cdots+p_n=p \\ p_{n-1}\neq p}} \underbrace{d_{p_1}(e_\alpha)\Big(\cdots d_{p_{n-3}}(e_\alpha)}_{(n-3)\text{ times}}\Big(d_{p_{n-2}}(r)\big(d_{p_{n-1}}(x)d_{p_n}(u)\big)\Big)\cdots\Big) \\
& - d_p\Big(\underbrace{e_\alpha\Big(\cdots e_\alpha}_{(n-3)\text{ times}}\big(r(yu)\big)\cdots\Big)\Big) \\
& + \sum_{\substack{p_1+\cdots+p_n=p \\ p_{n-1}\neq p}} \underbrace{d_{p_1}(e_\alpha)\Big(\cdots d_{p_{n-3}}(e_\alpha)}_{(n-3)\text{ times}}\Big(d_{p_{n-2}}(r)\big(d_{p_{n-1}}(y)d_{p_n}(u)\big)\Big)\cdots\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= d_p\Big(\underbrace{e_\alpha\Big(\cdots e_\alpha}_{(n-3)\text{ times}}\Big(r\big((x+y)u\big)\Big)\cdots\Big)\Big) - d_p\Big(\underbrace{e_\alpha\Big(\cdots e_\alpha}_{(n-3)\text{ times}}\big(r(xu)\big)\cdots\Big)\Big)\\
&\quad -d_p\Big(\underbrace{e_\alpha\Big(\cdots e_\alpha}_{(n-3)\text{ times}}\big(r(yu)\big)\cdots\Big)\Big)\\
&= d_p\Big(\underbrace{e_\alpha\Big(\cdots e_\alpha}_{(n-2)\text{ times}}\big(r(xu+yu)\big)\cdots\Big)\Big) - d_p\Big(\underbrace{e_\alpha\Big(\cdots e_\alpha}_{(n-2)\text{ times}}\big(r(xu)\big)\cdots\Big)\Big)\\
&\quad -d_p\Big(\underbrace{e_\alpha\Big(\cdots e_\alpha}_{(n-2)\text{ times}}\big(r(yu)\big)\cdots\Big)\Big)\\
&= \sum_{p_1+\cdots+p_n=p}\underbrace{d_{p_1}(e_\alpha)\Big(\cdots d_{p_{n-2}}(e_\alpha)}_{(n-2)\text{ times}}\big(d_{p_{n-1}}(r)d_{p_n}(xu+yu)\big)\cdots\Big)\\
&\quad -\sum_{p_1+\cdots+p_n=p}\underbrace{d_{p_1}(e_\alpha)\Big(\cdots d_{p_{n-2}}(e_\alpha)}_{(n-2)\text{ times}}\big(d_{p_{n-1}}(r)d_{p_n}(xu)\big)\cdots\Big)\\
&\quad -\sum_{p_1+\cdots+p_n=p}\underbrace{d_{p_1}(e_\alpha)\Big(\cdots d_{p_{n-2}}(e_\alpha)}_{(n-2)\text{ times}}\big(d_{p_{n-1}}(r)d_{p_n}(yu)\big)\cdots\Big)\\
&= \underbrace{e_\alpha\Big(\cdots e_\alpha}_{(n-2)\text{ times}}\big(rd_p(xu+yu)\big)\cdots\Big) - \underbrace{e_\alpha\Big(\cdots e_\alpha}_{(n-2)\text{ times}}\big(rd_p(xu)\big)\cdots\Big)\\
&\quad -\underbrace{e_\alpha\Big(\cdots e_\alpha}_{(n-2)\text{ times}}\big(rd_p(yu)\big)\cdots\Big)\\
&= \underbrace{e_\alpha\Big(\cdots e_\alpha}_{(n-2)\text{ times}}\Big(r\big(d_p(xu+yu)-d_p(xu)-d_p(yu)\big)\Big)\cdots\Big)\\
&= \underbrace{e_\alpha\Big(\cdots e_\alpha}_{(n-2)\text{ times}}\big(rf(xu,yu)\big)\cdots\Big)\\
&= e_\alpha\cdot rf(xu,yu),
\end{aligned}
$$

for all $\alpha \in \Lambda$. Hence, by a similar argument as in the proof of the case $n = 3$ we prove that $f(x,y)u = f(xu, yu)$. Similarly, we prove $uf(x,y) = f(ux, uy)$. We can conclude therefore that $f(x,y) = 0$ for all $x, y \in \mathfrak{R}$, by [2, Theorem 2.1]. Thus, the result is true for the positive integer p and holds the induction hypothesis. Therefore, by the mathematical induction principle, we can conclude that $d_p(x+y) = d_p(x) + d_p(y)$ for all $x, y \in \mathfrak{R}$ and for each $p \in \mathbb{N}$. The proof is complete. □

Corollary 2.1. *Let $\mathfrak{R}$ be a prime alternative ring containing a nontrivial idempotent. Then every n-multiplicative higher derivation $D = \{d_p\}_{p\in\mathbb{N}}$ of $\mathfrak{R}$ is additive.*

Corollary 2.2. *Let $\mathfrak{R}$ be an associative ring containing a family $\{e_\alpha | \alpha \in \Lambda\}$ of idempotents which satisfies:*

(i) If $x \in \mathfrak{R}$ is such that $x\mathfrak{R} = 0$, then $x = 0$;

(ii) If $x \in \mathfrak{R}$ is such that $e_\alpha \mathfrak{R} x = 0$ (or $e_\alpha \mathfrak{R} x = 0$) for all $\alpha \in \Lambda$, then $x = 0$ (and hence $\mathfrak{R}x = 0$ implies $x = 0$);

(iii) For each $\alpha \in \Lambda$ and $x \in \mathfrak{R}$, if $(e_\alpha x e_\alpha)\mathfrak{R}(1 - e_\alpha) = 0$ then $e_\alpha x e_\alpha = 0$.

Then every n-multiplicative higher derivation $D = \{d_p\}_{p\in\mathbb{N}}$ of $\mathfrak{R}$ is additive.

Corollary 2.3. *Let $\mathfrak{R}$ be a prime associative ring containing a nontrivial idempotent. Then every n-multiplicative higher derivation $D = \{d_p\}_{p\in\mathbb{N}}$ of $\mathfrak{R}$ is additive.*

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CHARACTERIZATION OF ADDITIVE n-MULTIPLICATIVE HIGHER DERIVATIONS OF NONASSOCIATIVE ALGEBRAS

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Abstract

In this paper we generalize Mirzavaziri's results on the characterization of higher derivations on nonassociative algebras.

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1 Introduction

Let $X = \{x_i\}_{i\in\mathbb{N}}$ be an arbitrary set of variables. A *nonassociative monomial of degree* 1 is any element of X. Given a natural number $n > 1$, a *nonassociative monomial of degree* n is an expression of the form $(u)(v)$, where u is a nonassociative monomial of some degree i and v a nonassociative monomial of degree $n - i$. A *nonassociative polynomial* p over a nonassociative algebra $\mathscr{A}$ over a field F is any formal additive combination of nonassociative monomials with coefficients in this field. If p includes no variables except $x_1, \ldots, x_n$ and $a_1, \ldots, a_n$ is a set of elements of $\mathscr{A}$, then $p(a_1, \ldots, a_n)$ is an element of $\mathscr{A}$ which results by applying the sequence of operations forming p to $a_1, \ldots, a_n$ in place of $x_1, \ldots, x_n$.

Let $\mathscr{A}$ be a nonassociative algebra over a field F of characteristic zero and n be a positive integer. A mapping $d : \mathscr{A} \to \mathscr{A}$ is called a *n-multiplicative derivation of $\mathscr{A}$ by a nonassociative monomial $m = m(x_1, \ldots, x_n)$ of degree n* (resp., *n-multiplicative derivation of $\mathscr{A}$*) if

$$d\big(m(a_1, \cdots, a_n)\big) = \sum_{i=1}^{n} m\big(a_1, \cdots, d(a_i), \cdots, a_n\big)$$

for arbitrary elements $a_1, \cdots, a_n \in \mathscr{A}$ (resp., for all nonassociative monomial $m = m(a_1, \cdots, a_n)$ of degree n and arbitrary elements $a_1, \cdots, a_n \in \mathscr{A}$). If $d(a_1a_2) = d(a_1)a_2 + a_1d(a_2)$ for all $a_1, a_2 \in \mathscr{A}$, we just say that d is a *multiplicative derivation* of $\mathscr{A}$.

A set $D = \{d_p\}_{p\geq 0}$ of mappings $d_p : \mathscr{A} \to \mathscr{A}$ is called a *n-multiplicative higher derivation of $\mathscr{A}$ by a nonassociative monomial $m = m(x_1, \ldots, x_n)$ of degree n* (resp., *n-multiplicative higher derivation of $\mathscr{A}$*), if d_0 is the identity mapping I of $\mathscr{A}$ and

$$d_p\big(m(a_1, \cdots, a_n)\big) = \sum_{p_1+\cdots+p_n=p} m\big(d_{p_1}(a_1), \cdots, d_{p_n}(a_n)\big)$$

for arbitrary elements $a_1, \cdots, a_n$ of $\mathscr{A}$ and each non-negative integer p (resp., for all nonassociative monomial $m = m(a_1, \cdots, a_n)$ of degree n, arbitrary elements $a_1, \cdots, a_n$ of $\mathscr{A}$ and each non-negative integer p). If d_0 is the identity mapping I of $\mathscr{A}$ and $d_p(a_1a_2) = \sum_{p_1+p_2=p} d_{p_1}(a_1)d_{p_2}(a_2)$, for

arbitrary elements a_1, a_2 of $\mathscr{A}$ and each non-negative integer p, we just say that d is a *multiplicative higher derivation* of $\mathscr{A}$.

A mapping $d : \mathscr{A} \to \mathscr{A}$ is called *additive* if $d(a_1 + a_2) = d(a_1) + d(a_2)$, for all $a_1, a_2 \in \mathscr{A}$. Certainly, any linear mapping is additive.

A n-multiplicative higher derivation of $\mathscr{A}$ by a nonassociative monomial $m = m(x_1, \ldots, x_n)$ of degree n (resp., n-multiplicative higher derivation of $\mathscr{A}$) $D = \{d_p\}_{p\geq 0}$ is called *additive* if the mapping d_p is additive, for each non-negative integer p. A linear multiplicative derivation of $\mathscr{A}$ is called a *derivation of* $\mathscr{A}$ and a linear multiplicative higher derivation of $\mathscr{A}$ is called a *higher derivation of* $\mathscr{A}$.

Throughout this article $\mathscr{A}$ denotes a nonassociative algebra over a field of characteristic zero and I is the identity mapping on $\mathscr{A}$.

The author in [3] characterized the higher derivations of an algebra $\mathscr{A}$ and established a one to one correspondence between this class with the set of all sequences of derivations of $\mathscr{A}$. His main results were as follows:

Theorem 1.1. *[3, Theorem 2.3.] Let $\{d_p\}_{p\geq 0}$ be a higher derivation on an algebra $\mathscr{A}$ with $d_0 = I$. Then there is a sequence $\{\delta_p\}_{p\geq 0}$ of derivations of $\mathscr{A}$ such that*

$$d_p = \sum_{i=1}^{p} \left(\sum_{\sum_{j=1}^{i} r_j = p} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) \delta_{r_1} \cdots \delta_{r_i} \right),$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = p$,

and

Theorem 1.2. *[3, Theorem 2.5.] Let $\mathscr{A}$ be an algebra. D the set of all higher derivations $\{d_p\}_{p\geq 0}$ on $\mathscr{A}$ with $d_0 = I$ and Δ the set of all sequences $\{\delta_p\}_{p\geq 0}$ of derivations on $\mathscr{A}$ with $\delta_0 = 0$. Then, there is a one to one correspondence between D and Δ.*

Studies on the characterization of the additivity of n-multiplicative derivations and n-multiplicative higher derivations on classes of associative and nonassociative rings is starting to catch the attention of mathematicians. In

particular, the author in [5] characterized the additivity of the n-multiplicative derivations on the class of associative rings $\mathscr{R}$ containing a family of nontrivial idempotents satisfying the Martindale's conditions and the authors in [2] characterized the additivity of the n-multiplicative derivations on the class of nonassociative rings $\mathscr{R}$ similar, namely: the alternative rings containing a family of nontrivial idempotents satisfying the Martindale's conditions. Also, the authors in [1] characterized the additivity of n-multiplicative higher derivations of the class of associative and alternative rings $\mathscr{R}$, similar to the aforementioned cases. However, we note that although there are already some results on this topic, the characterization of the additivity of n-multiplicative derivations and of n-multiplicative higher derivations for a wide classes of these rings is far from being resolved.

In this paper we generalize the Theorems 1.1 and 1.2 for the class of additive n-multiplicative higher derivations of an arbitrary nonassociative algebra $\mathscr{A}$ by an arbitrary nonassociative monomial of degree n.

2 The results

Proposition 2.1. *Let $\mathscr{A}$ be a nonassociative algebra and $D = \{d_p\}_{p\geq 0}$ be an additive n-multiplicative higher derivation of $\mathscr{A}$ by a nonassociative monomial m of degree n with $d_0 = I$. Then, there is a unique sequence $\{\delta_p\}_{p\geq 0}$ of additive n-multiplicative derivations of $\mathscr{A}$ by the same nonassociative monomial m with $\delta_0 = 0$ and such that*

$$(p+1)d_{p+1} = \sum_{k=0}^{p} \delta_{k+1} d_{p-k},$$

for all non-negative integer p.

Proof. We start taking $\delta_0 = 0$. Next, using the mathematical induction on p, observe that for $p = 0$, from the hypotheses of the theorem, we have

$$\begin{aligned} & d_1\big(m(a_1, \cdots, a_n)\big) \\ = & \sum_{p_1+\cdots+p_n=1} m\big(d_{p_1}(a_1), \cdots, d_{p_n}(a_n)\big) \end{aligned}$$

$$= \sum_{i=1}^{n} m(a_1, \cdots, d_1(a_i), \cdots, a_n)$$

for all elements $a_1, \cdots, a_n$ of $\mathscr{A}$. This results that if we define $\delta_1 = d_1$, then δ_1 is an additive n-multiplicative derivation of $\mathscr{A}$ by the same nonassociative monomial m and $(0+1)d_{0+1} = d_1 = \delta_1 = \sum_{k=0}^{0} \delta_{k+1} d_{0-k}$. Now, assume that for some positive integer p there are additive n-multiplicative derivations of $\mathscr{A}$ by the same nonassociative monomial m $\delta_1, \cdots, \delta_p$ such that

$$(q+1)d_{q+1} = \sum_{k=0}^{q} \delta_{k+1} d_{q-k},$$

for every $q = 0, 1, \cdots, p-1$. Let us define the mapping $\delta_{p+1} : \mathscr{A} \to \mathscr{A}$ by

$$\delta_{p+1} = (p+1)d_{p+1} - \sum_{k=0}^{p-1} \delta_{k+1} d_{p-k}. \tag{1}$$

It is easy to verify that δ_{p+1} is an additive mapping. It follows that, for arbitrary elements $a_1, \cdots, a_n$ of $\mathscr{A}$, we have

$$\begin{aligned}
&\delta_{p+1}\big(m(a_1, \cdots, a_n)\big) \\
= \;& (p+1)d_{p+1}\big(m(a_1, \cdots, a_n)\big) - \sum_{k=0}^{p-1} \delta_{k+1} d_{p-k}\big(m(a_1, \cdots, a_n)\big) \\
= \;& (p+1) \sum_{p_1+\cdots+p_n=p+1} m\big(d_{p_1}(a_1), \cdots, d_{p_n}(a_n)\big) \\
& - \sum_{k=0}^{p-1} \delta_{k+1}\Big(\sum_{p_1+\cdots+p_n=p-k} m\big(d_{p_1}(a_1), \cdots, d_{p_n}(a_n)\big)\Big) \\
= \;& (p+1) \sum_{p_1+\cdots+p_n=p+1} m\big(d_{p_1}(a_1), \cdots, d_{p_n}(a_n)\big) \\
& - \sum_{k=0}^{p-1} \sum_{p_1+\cdots+p_n=p-k} \delta_{k+1}\big(m\big(d_{p_1}(a_1), \cdots, d_{p_n}(a_n)\big)\big) \\
= \;& (p_1 + \cdots + p_n) \sum_{p_1+\cdots+p_n=p+1} m\big(d_{p_1}(a_1), \cdots, d_{p_n}(a_n)\big)
\end{aligned}$$

$$-\sum_{l=1}^{n}\Big(\sum_{k=0}^{p-1}\sum_{p_1+\cdots+p_n=p-k} m\big(d_{p_1}(a_1),\cdots,\delta_{k+1}(d_{p_l}(a_l)),\cdots,d_{p_n}(a_n)\big)\Big).$$

Set

$$\begin{aligned} & K_l\big(m(a_1,\cdots,a_n)\big) \\ = \; & p_l \sum_{p_1+\cdots+p_n=p+1} m\big(d_{p_1}(a_1),\cdots,d_{p_n}(a_n)\big) \\ & -\sum_{k=0}^{p-1}\sum_{p_1+\cdots+p_n=p-k} m\big(d_{p_1}(a_1),\cdots,\delta_{k+1}(d_{p_l}(a_l)),\cdots,d_{p_n}(a_n)\big), \end{aligned}$$

for every integer l $(1 \le l \le n)$. Certainly, holds $\delta_{p+1}\big(m(a_1,\cdots,a_n)\big) = \sum_{l=1}^{n} K_l\big(m(a_1,\cdots,a_n)\big)$. Let us prove that

$$K_l = m\big(a_1,\cdots,\delta_{p+1}(a_l),\cdots,a_n\big),$$

for every integer l $(1 \le l \le n)$. However, we only consider the proof for the parcel $K_1\big(m(a_1,\cdots,a_n)\big)$, namely:

$$\begin{aligned} & K_1\big(m(a_1,\cdots,a_n)\big) \\ = \; & p_1 \sum_{p_1+p_2+\cdots+p_n=p+1} m\big(d_{p_1}(a_1),d_{p_2}(a_2),\cdots,d_{p_n}(a_n)\big) \qquad (2) \\ & -\sum_{k=0}^{p-1}\sum_{p_1+\cdots+p_n=p-k} m\big(\delta_{k+1}(d_{p_1}(a_1)),d_{p_2}(a_2),\cdots,d_{p_n}(a_n)\big), \qquad (3) \end{aligned}$$

since the proof of the remaining parcels are similar. Thus, take the parameter $p_2+\cdots+p_n = l$ $(0 \le l \le p+1)$, in the summation in (2). It follows that $p_1 = p-l+1$ and we can write (2) as follows

$$\begin{aligned} & p_1 \sum_{p_1+p_2+\cdots+p_n=p+1} m\big(d_{p_1}(a_1),d_{p_2}(a_2),\cdots,d_{p_n}(a_n)\big) \\ = \; & (p-l+1)\sum_{l=0}^{p+1}\sum_{p_2+\cdots+p_n=l} m\big(d_{p-l+1}(a_1),d_{p_2}(a_2),\cdots,d_{p_n}(a_n)\big) \end{aligned}$$

$$= (p+1)m\big(d_{p+1}(a_1), a_2, \cdots, a_n\big) + (p-l+1)\sum_{l=1}^{p}\sum_{p_2+\cdots+p_n=l} m\big(d_{p-l+1}(a_1), d_{p_2}(a_2), \cdots, d_{p_n}(a_n)\big). \quad (4)$$

Next, opening the second summation of K_1, in (3), we have

$$\sum_{k=0}^{p-1}\sum_{p_1+p_2+\cdots+p_n=p-k} m\big(\delta_{k+1}(d_{p_1}(a_1)), d_{p_2}(a_2), \cdots, d_{p_n}(a_n)\big)$$

$$= \begin{cases} \displaystyle\sum_{p_1+p_2+\cdots+p_n=p} m\big(\delta_1(d_{p_1}(a_1)), d_{p_2}(a_2), \cdots, d_{p_n}(a_n)\big) \\ + \displaystyle\sum_{p_1+p_2+\cdots+p_n=p-1} m\big(\delta_2(d_{p_1}(a_1)), d_{p_2}(a_2), \cdots, d_{p_n}(a_n)\big) \\ \vdots \\ + \displaystyle\sum_{p_1+p_2+\cdots+p_n=p-k} m\big(\delta_{k+1}(d_{p_1}(a_1)), d_{p_2}(a_2), \cdots, d_{p_n}(a_n)\big) \\ \vdots \\ + \displaystyle\sum_{p_1+p_2+\cdots+p_n=2} m\big(\delta_{p-1}(d_{p_1}(a_1)), d_{p_2}(a_2), \cdots, d_{p_n}(a_n)\big) \\ + \displaystyle\sum_{p_1+p_2+\cdots+p_n=1} m\big(\delta_p(d_{p_1}(a_1)), d_{p_2}(a_2), \cdots, d_{p_n}(a_n)\big). \end{cases} \quad (5)$$

Then, take the parameter $p_2+\cdots+p_n = l$ $(0 \leq l \leq p)$ and consider all parcels which are part of each of the summations in (5), indicate by this parameter. It follows that we can rewrite (5) by regrouping these parcels into new classes, determined by the parameter l. Thus, consider the parcels whose are indicate by $p_2+\cdots+p_n = 0$. In this case, all summations have such parcels, satisfying this condition. Taking these parcels and their respective sum we get

$$m\big(\delta_1(d_p(a_1)), a_2, \cdots, a_n\big) + m\big(\delta_2(d_{p-1}(a_1)), a_2, \cdots, a_n\big) + \cdots$$
$$+ m\big(\delta_{k+1}(d_{p-k}(a_1)), a_2, \cdots, a_n\big) + \cdots + m\big(\delta_{p-1}(d_2(a_1)), a_2, \cdots, a_n\big)$$
$$+ m\big(\delta_p(d_1(a_1)), a_2, \cdots, a_n\big)$$

$$
\begin{aligned}
&= m\Big(\sum_{k=0}^{p-1}\delta_{k+1}(d_{p-k}(a_1)), a_2, \cdots, a_n\Big)\\
&= m\big((p+1)d_{p+1}(a_1) - \delta_{p+1}(a_1), a_2, \cdots, a_n\big) \text{ (by (1))}\\
&= m\big((p+1)d_{p+1}(a_1), a_2, \cdots, a_n\big) - m\big(\delta_{p+1}(a_1), a_2, \cdots, a_n\big). \qquad (6)
\end{aligned}
$$

Now, consider the parcels whose are indicate by parameter $p_2 + \cdots + p_n = l$ $(1 \leq l \leq p)$. Note that in this case only the first $p - l + 1$ summations, in (5), have such parcels indicate by this parameter. Taking these parcels in their respective summations and their sum, results that

$$
\begin{aligned}
&\sum_{p_2+\cdots+p_n=l} m\big(\delta_1(d_{p-l}(a_1)), d_{p_2}(a_2), \cdots, d_{p_n}(a_n)\big)\\
&+ \sum_{p_2+\cdots+p_n=l} m\big(\delta_2(d_{p-l-1}(a_1)), d_{p_2}(a_2), \cdots, d_{p_n}(a_n)\big)\\
&\vdots\\
&+ \sum_{p_2+\cdots+p_n=l} m\big(\delta_{k+1}(d_{p-l-k}(a_1)), d_{p_2}(a_2), \cdots, d_{p_n}(a_n)\big)\\
&\vdots\\
&+ \sum_{p_2+\cdots+p_n=l} m\big(\delta_{p-l}(d_1(a_1)), d_{p_2}(a_2), \cdots, d_{p_n}(a_n)\big)\\
&+ \sum_{p_2+\cdots+p_n=l} m\big(\delta_{p-l+1}(a_1), d_{p_2}(a_2), \cdots, d_{p_n}(a_n)\big)\\
&= \sum_{p_2+\cdots+p_n=l} m\Big(\sum_{k=0}^{p-l}\delta_{k+1}(d_{p-l-k}(a_1)), d_{p_2}(a_2), \cdots, d_{p_n}(a_n)\Big)\\
&= (p-l+1)\sum_{p_2+\cdots+p_n=l} m\big(d_{p-l+1}(a_1), d_{p_2}(a_2), \cdots, d_{p_n}(a_n)\big). \qquad (7)
\end{aligned}
$$

By (6) and (7), the summation in (3) can be rewrite as

$$= \begin{cases} m\big((p+1)d_{p+1}(a_1), a_2, \cdots, a_n\big) \\ + (p-l+1)\sum_{l=1}^{p} \sum_{p_2+\cdots+p_n=l} m\big(d_{p-l+1}(a_1), d_{p_2}(a_2), \cdots, d_{p_n}(a_n)\big) \\ - m\big(\delta_{p+1}(a_1), a_2, \cdots, a_n\big). \end{cases} \tag{8}$$

From (4) and (8) we conclude that

$$K_1\big(m(a_1, a_2, \cdots, a_n)\big) = m\big(\delta_{p+1}(a_1), a_2, \cdots, a_n\big).$$

Finally, it is easy to see that the sequence is uniquely determined. With this the theorem is proved. □

Now we are able to prove the analogues of Theorems 1.1 and 1.2, for the class of additive n-multiplicative higher derivations of $\mathscr{A}$ by a nonassociative monomial m. In the first theorem what follows, our proof is similar to that in [3].

Theorem 2.1. *Let $\mathscr{A}$ be a nonassociative algebra and $D = \{d_p\}_{p\geq 0}$ be a additive n-multiplicative higher derivation of $\mathscr{A}$ by a nonassociative monomial m with $d_0 = I$. Then, there is a sequence $\{\delta_p\}_{p\geq 0}$ of additive n-multiplicative derivation of $\mathscr{A}$ by the same nonassociative monomial m such that $\delta_0 = 0$ and*

$$d_p = \sum_{i=1}^{p} \left(\sum_{\sum_{j=1}^{i} r_j = p} \Big(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \Big) \delta_{r_1} \cdots \delta_{r_i} \right) \tag{9}$$

on $\mathscr{A}$ and for each non-negative integer p, where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = p$.

Proof. By Proposition 2.1, there is a unique sequence $\{\delta_p\}_{p\geq 0}$ of additive n-multiplicative derivations of $\mathscr{A}$ by the same nonassociative monomial m such that $\delta_0 = 0$ and

$$(p+1)d_{p+1} = \sum_{k=0}^{p} \delta_{k+1} d_{p-k},$$

for all non-negative integer p. Now, let us define $a_{r_1,\cdots,r_i} = \prod_{j=1}^{i} \frac{1}{r_j+\cdots+r_i}$ and note that if $r_1+\cdots+r_i = p+1$, then $(p+1)a_{r_1,\cdots,r_i} = a_{r_2,\cdots,r_i}$ and that $a_{p+1} = \frac{1}{p+1}$. It follows that

$$\begin{aligned}
&\sum_{i=1}^{p+1}\left(\sum_{\sum_{j=1}^{i} r_j=p+1}\left(\prod_{j=1}^{i}\frac{1}{r_j+\cdots+r_i}\right)\delta_{r_1}\cdots\delta_{r_i}\right)\\
&= \sum_{i=2}^{p+1}\left(\sum_{\sum_{j=1}^{i} r_j=p+1} a_{r_1,\cdots,r_i}\delta_{r_1}\cdots\delta_{r_i}\right)+\frac{1}{p+1}\delta_{p+1}\\
&= \frac{1}{p+1}\left(\sum_{i=2}^{p+1}\left(\sum_{\sum_{j=1}^{i} r_j=p+1}(p+1)a_{r_1,\cdots,r_i}\delta_{r_1}\cdots\delta_{r_i}\right)+\delta_{p+1}\right)\\
&= \frac{1}{p+1}\left(\sum_{i=2}^{p+1}\left(\sum_{r_1=1}^{p+2-i}\delta_{r_1}\sum_{\sum_{j=2}^{i} r_j=p+1-r_1} a_{r_2,\cdots,r_i}\delta_{r_2}\cdots\delta_{r_i}\right)+\delta_{p+1}\right)\\
&= \frac{1}{p+1}\left(\sum_{r_1=1}^{p}\delta_{r_1}\sum_{i=2}^{p-(r_1-1)}\left(\sum_{\sum_{j=2}^{i} r_j=p-(r_1-1)} a_{r_2,\cdots,r_i}\delta_{r_2}\cdots\delta_{r_i}\right)+\delta_{p+1}\right)\\
&= \frac{1}{p+1}\left(\sum_{r_1=1}^{p}\delta_{r_1}d_{p-(r_1-1)}+\delta_{p+1}\right)\\
&= \frac{1}{p+1}\left(\sum_{k=0}^{p}\delta_{k+1}d_{p-k}\right)\\
&= d_{p+1},
\end{aligned}$$

on $\mathscr{A}$ and for each non-negative integer p. □

Theorem 2.2. *Let $\mathscr{A}$ be a nonassociative algebra, $D(n;m)$ be the set of all additive n-multiplicative higher derivations of $\mathscr{A}$ by a nonassociative monomial m $D = \{d_p\}_{p\geq 0}$ with $d_0 = I$ and $\Delta(n;m)$ be the set of all sequences $\Delta = \{\delta_p\}_{p\geq 0}$ of additive n-multiplicative derivations of $\mathscr{A}$ by the same nonassociative monomial m with $\delta_0 = 0$. Then, there is a one-to-one correspondence between $\Delta(n;m)$ and $D(n;m)$.*

Proof. Let $\Delta = \{\delta_p\}_{p\geq 0}$ be a sequence of additive n-multiplicative derivations of $\mathscr{A}$ by a nonassociative monomial m of degree n with $\delta_0 = 0$ and define the sequence $D = \{d_p\}_{p\geq 0}$ of mappings $d_p : \mathscr{A} \to \mathscr{A}$, for every non-negative integer p, by $d_0 = I$ and

$$d_p = \sum_{i=1}^{p} \left(\sum_{\sum_{j=1}^{i} r_j = p} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) \delta_{r_1} \cdots \delta_{r_i} \right),$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = p$. Let us show that $D = \{d_p\}_{p\geq 0}$ is an additive n-multiplicative higher derivation of $\mathscr{A}$ by the nonassociative monomial m. First, we observe that $D = \{d_p\}_{p\geq 0}$ satisfies the recursive relation

$$(p+1)d_{p+1} = \sum_{k=0}^{p} \delta_{k+1} d_{p-k},$$

for all non-negative integer p, by Theorem 2.1. Thus, by mathematical induction on p, we start with $d_0\big(m(a_1, \cdots, a_n)\big) = m(a_1, \cdots, a_n) = \sum_{p_1+\cdots+p_n=0} m\big(d_{p_1}(a_1) \cdots d_{p_n}(a_n)\big)$. Now, assume that for some positive integer p holds $d_l\big(m(a_1, \cdots, a_n)\big) = \sum_{p_1+\cdots+p_n=l} m\big(d_{p_1}(a_1), \cdots, d_{p_n}(a_n)\big)$, for all non-negative integer $l \leq p$. It follows that

$$\begin{aligned}
&(p+1)d_{p+1}\big(m(a_1, a_2, \cdots, a_n)\big) \\
&= \sum_{k=0}^{p} \delta_{k+1} d_{p-k}\big(m(a_1, a_2, \cdots, a_n)\big) \\
&= \sum_{k=0}^{p} \delta_{k+1}\Big(\sum_{p_1+\cdots+p_n=p-k} m\big(d_{p_1}(a_1), \cdots, d_{p_n}(a_n)\big)\Big) \\
&= \sum_{k=0}^{p} \sum_{p_1+\cdots+p_n=p-k} \delta_{k+1}\big(m\big(d_{p_1}(a_1), \cdots, d_{p_n}(a_n)\big)\big) \\
&= \sum_{k=0}^{p} \sum_{p_1+\cdots+p_i+\cdots+p_n=p-k} \Big(\sum_{i=1}^{n} m\big(d_{p_1}(a_1), \cdots, \delta_{k+1}\big(d_{p_i}(a_i)\big), \\
&\qquad\qquad \cdots, d_{p_n}(a_n)\big)\Big)
\end{aligned}$$

$$= \sum_{i=1}^{n}\Big(\sum_{k=0}^{p}\sum_{p_1+\cdots+p_i+\cdots+p_n=p-k} m\big(d_{p_1}(a_1),\cdots,\delta_{k+1}\big(d_{p_i}(a_i)\big),\cdots,d_{p_n}(a_n)\big)\Big).$$

Let us show that

$$\sum_{k=0}^{p}\sum_{p_1+\cdots+p_i+\cdots+p_n=p-k} m\big(d_{p_1}(a_1),\cdots,\delta_{k+1}\big(d_{p_i}(a_i)\big),\cdots,d_{p_n}(a_n)\big) \tag{10}$$

$$= \sum_{l=0}^{p}(p-l+1)\sum_{p_2+\cdots+p_n=l} m\big(d_{p_1}(a_1),\cdots,d_{p-l+1}(a_i),\cdots,d_{p_n}(a_n)\big),$$

for every i $(1 \le i \le n)$. As in the Theorem 2.1, the proof is made only for case $i = 1$, since the proof of the remaining cases are similar. Thus, write (10) in the form

$$= \begin{cases} \displaystyle\sum_{p_1+p_2+\cdots+p_n=p} m\big(\delta_1\big(d_{p_1}(a_1)\big),d_{p_2}(a_2),\cdots,d_{p_n}(a_n)\big) \\ \vdots \\ + \displaystyle\sum_{p_1+p_2+\cdots+p_n=p-k} m\big(\delta_{k+1}\big(d_{p_1}(a_1)\big),d_{p_2}(a_2),\cdots,d_{p_n}(a_n)\big) \\ \vdots \\ + \displaystyle\sum_{p_1+p_2+\cdots+p_n=1} m\big(\delta_p\big(d_{p_1}(a_1)\big),d_{p_2}(a_2),\cdots,d_{p_n}(a_n)\big) \\ + \displaystyle\sum_{p_1+p_2+\cdots+p_n=0} m\big(\delta_{p+1}\big(d_{p_1}(a_1)\big),d_{p_2}(a_2),\cdots,d_{p_n}(a_n)\big). \end{cases} \tag{11}$$

Consider in the summations, in (11), the parcels whose are indicate by $p_2,\cdots,p_n$ such that $p_2+\cdots+p_n = 0$. Note that all summations have such parcels, satisfying this condition. Taking the sum of them we obtain

$$m\big(\delta_1\big(d_p(a_1)\big),a_2,\cdots,a_n\big)+\cdots+m\big(\delta_{k+1}\big(d_{p-k}(a_1)\big),a_2,\cdots,a_n\big)$$

$$
\begin{aligned}
&+\cdots+m\big(\delta_p\big(d_1(a_1)\big),a_2,\cdots,a_n\big)+m\big(\delta_{p+1}(a_1),a_2,\cdots,a_n\big)\\
=\;& m\Big(\sum_{k=0}^{p-1}\delta_{k+1}\big(d_{p-k}(a_1)\big)+\delta_{p+1}(a_1),a_2,\cdots,a_n\Big)\\
=\;& (p+1)m\big(d_{p+1}(a_1),a_2,\cdots,a_n\big). \qquad (12)
\end{aligned}
$$

Now, consider the parcels whose are indicate by $p_2,\cdots,p_n$ such that $p_2+\cdots+p_n=l$ $(1\le l\le p)$. Note that in this case only the first $p-l+1$ summations have such parcels satisfying this condition. Taking the sum of them, again, we obtain

$$
\begin{aligned}
&\sum_{p_2+\cdots+p_n=l} m\big(\delta_1\big(d_{p-l}(a_1)\big),d_{p_2}(a_2),\cdots,d_{p_n}(a_n)\big)\\
&\vdots\\
&+\sum_{p_2+\cdots+p_n=l} m\big(\delta_{k+1}\big(d_{p-l-k}(a_1)\big),d_{p_2}(a_2),\cdots,d_{p_n}(a_n)\big)\\
&\vdots\\
&+\sum_{p_2+\cdots+p_n=l} m\big(\delta_{p-l}\big(d_1(a_1)\big),d_{p_2}(a_2),\cdots,d_{p_n}(a_n)\big)\\
&+\sum_{p_2+\cdots+p_n=l} m\big(\delta_{p-l+1}(a_1),d_{p_2}(a_2),\cdots,d_{p_n}(a_n)\big)\\
=\;& m\Big(\sum_{k=0}^{p-l}\delta_{k+1}\big(d_{p-l-k}(a_1)\big),d_{p_2}(a_2),\cdots,d_{p_n}(a_n)\Big)\\
=\;& \sum_{p_2+\cdots+p_n=l}(p-l+1)m\big(d_{p-l+1}(a_1),d_{p_2}(a_2),\cdots,d_{p_n}(a_n)\big). \qquad (13)
\end{aligned}
$$

From (12) and (13) follows that

$$
\begin{aligned}
&\sum_{k=0}^{p}\sum_{p_1+p_2+\cdots+p_n=p-k} m\big(\delta_{k+1}\big(d_{p_1}(a_1)\big),d_{p_2}(a_2),\cdots,d_{p_n}(a_n)\big)\\
=\;& (p-l+1)\sum_{l=0}^{p}\sum_{p_2+\cdots+p_n=l} m\big(d_{p-l+1}(a_1),d_{p_2}(a_2),\cdots,d_{p_n}(a_n)\big). \qquad (14)
\end{aligned}
$$

Therefore we conclude that

$$\begin{aligned}
&(p+1)d_{p+1}\big(m(a_1,a_2,\cdots,a_n)\big)\\
&= \sum_{i=1}^{n}(p-l+1)\sum_{l=0}^{p}\sum_{p_1+\cdots+p_{i-1}+p_{i+1}\cdots+p_n=l} m\big(d_{p_1}(a_1),\\
&\qquad\qquad \cdots,d_{p-l+1}(a_i),\cdots,d_{p_n}(a_n)\big)\\
&= \sum_{i=1}^{n}(p-l+1)\sum_{l=0}^{p+1}\sum_{p_1+\cdots+p_{i-1}+p_{i+1}\cdots+p_n=l} m\big(d_{p_1}(a_1),\\
&\qquad\qquad \cdots,d_{p-l+1}(a_i),\cdots,d_{p_n}(a_n)\big)\\
&= \sum_{i=1}^{n}p_i\sum_{p_1+\cdots+p_i+\cdots+p_n=p+1} m\big(d_{p_1}(a_1),\cdots,d_{p_i}(a_i),\cdots,d_{p_n}(a_n)\\
&= (p+1)\sum_{p_1+\cdots+p_i+\cdots+p_n=p+1} m\big(d_{p_1}(a_1),\cdots,d_{p_i}(a_i),\cdots,d_{p_n}(a_n).
\end{aligned}$$

This show that d_{p+1} is a additive n-multiplicative higher derivation of $\mathscr{A}$ by the same nonassociative monomial m. Consequently, the mathematical induction process on p allows us to conclude that $D = \{d_p\}_{p\geq 0}$ belongs to $D(n,m)$. Conversely, suppose that $D = \{d_p\}_{p\geq 0}$ is an additive n-multiplicative higher derivation of $\mathscr{A}$ by the same nonassociative monomial m and define the sequence of mappings $\{\delta_p\}_{p\geq 0}$, $\delta_p : \mathscr{A} \to \mathscr{A}$, for every non-negative integer p, by $\delta_0 = 0$ and $\delta_p = pd_p - \sum_{k=0}^{p-2}\delta_{k+1}d_{p-1-k}$. By Proposition 2.1 we have that $\{\delta_p\}_{p\geq 0}$ is the only sequence belonging to $\Delta(n,m)$ and satisfying such condition. It follows that the mapping $\varphi : \Delta(n,m) \to D(n,m)$ defined by $\varphi\big(\{\delta_p\}_{p\geq 0}\big) = \{d_p\}_{p\geq 0}$, where d_p is given by (9), for each non-negative integer p, is clearly a bijective mapping. □

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